Exercises DataBase
for the first year class in Mathematics
in Scuola Normale Superiore

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Collections of exercises and theoretical bits proposed to first year students in "Scuola Normale Superiore".

[this is a draft there are many errors]

§1 Introduction [100]

The Scuola Normale Superiore (SNS) is a prestigious university institution with special status that welcomes students into two distinct paths: the undergraduate course (parallel to undergraduate and master's degree programs) and the advanced course (PhD).

SNS students in the undergraduate course are required to take some ”internal courses” during the academic year, in addition to their regular university courses (to which they are duly enrolled at the University of Pisa). Every year, first-year SNS students in subjects such as Mathematics, Physics, Chemistry, and Biology have followed an annual internal Mathematics course. This course aims to delve deeper into and expand upon the traditional concepts included in the curricula of university courses that SNS students simultaneously attend.

Over the last 15 years, this course has covered several fundamental topics. It begins with a more thorough treatment of the foundations of Mathematics, including set theory based on the Zermelo-Fraenkel axioms, the construction of the set of natural numbers, and the characterization of real numbers as a complete ordered field. It then progresses to topics such as series and sequences, metric spaces and topology, differential calculus, and ordinary differential equations.

In these years, professors Giuseppe Da Prato, Fulvio Ricci, Luigi Ambrosio and Franco Flandoli have held the course. In addition to the author of this volume, Francesco Bonsante, Carlo Mantegazza, Simone Di Marino, Tommaso Pacini, Luciano Mari, Lorenzo Mazzieri, Andreas Hochenegger, Andrea Ferraguti, Alessandra Caraceni collaborated with TA.

The course notes have been published in [2].

The author has collaborated as TA, for more than ten years, accumulating a significant amount of theoretical material and exercises, which are now presented in this volume.

As is the case for text [2], this volume is not entirely self-contained as it is intended as a supplement to standard university courses in the first year. However, the first part is an exception because courses covering topics in logic fundamentals are not typically offered in the first year, and the texts used are often not written in a readily accessible language for first-year students. Therefore, Chapters 3 and 4 have been expanded to include the necessary theoretical elements, often disguised as exercises. Starting from Chapter 5, useful references to tackle the exercises are provided, along with some definitions and lemmas.

It should be noted that the numbering system in this volume follows a specific method: sections (and subsections), footnotes, and figures are numbered independently, while everything else in the volume follows a unique numbering system, divided by sections. This includes theorems, propositions, lemmas, equations, and more. The different numberings are made distinguishable by the use of Roman or Arabic numerals and/or special symbols, such as § for sections and † for notes.

ColDoc [2G2]

The author has also developed a software package called `ColDoc` (freely available) that facilitates the management of complex LaTeXdocuments and allows for online access. It can be used on both computers and mobile devices such as tablets and smartphones.

The `ColDoc` version of this text is accessible at https://coldoc.sns.it/CD/EDB; this platform was initially created to enhance interaction with students during the Covid pandemic. It has evolved in a featureful document system.

The ColDoc system divides the text into small elements, each identified by a UUID code. The UUID code permanently identifies an object; whereas the LaTeX system's assigned number could change if additional material were added before that object (e.g., in a future edition). Therefore, the UUID code can be used for bibliographic references and for making notes on an item of interest to share with colleagues or students, as the UUID code can be used to retrieve the item in the web interface. For example, this introduction can be found at https://coldoc.sns.it/UUID/EDB/2G1.

The ColDoc system also implements a multilingual LaTeX document management system: for instance, this text is available in both English and Italian.

Copyright [009]

§2 Notations [00B]

- $\mathbb{N}$ are the natural numbers, including zero.
- $\mathbb{Z}$ are the integers.
- $\mathbb{Q}$ are the rational numbers.
- $\mathbb{R}$ is the real line.
- $\mathbb{C}$ are the complex numbers.

A list of symbols is also available at the start of the index.

Remark 2.1. *The symbols $\wedge$ and $\vee$ can be used in two different contexts, where they assume different meanings.* [2DM]

- *If $x, y \in \mathbb{R}$ are real numbers, then $x \wedge y$ is the minimum of the two numbers, while $x \vee y$ is the maximum of the two numbers. This meaning is also appropriate when x, y are in a totally ordered set.* [†1]
- *In mathematical logic, $\wedge$ is the* conjunction *and $\vee$ is the* disjunction*. See 3.a.4.*

Remark 2.2. *The parentheses symbols () are unfortunately quite overloaded in common Mathematical language.* [2FG]

- *They are used to group algebraic operations, to induce a different order of operations (*wrt *the standard rules of precedence). For example, for $x, y \in \mathbb{R}$,* [†2] *the expression $x(y + 2)$ is identical to $xy + 2x$ and not to $xy + 2$.*
- *They are used to denote arguments of functions. For example the expression $f(x + y)$ should be read as $fx + fy$, if $f, x, y \in \mathbb{R}$* [†3]*; whereas, if f is a function $f : \mathbb{R} \to B$, then $f(x + y)$ is the result $f(z)$ obtained by evaluating f on the element $z = x + y$.*

  To distinguish these two usages, it may be sufficient to add an explicit symbol to denote "multiplication", writing $f*(x+y)$ when it should be read as $f*x+f*y$. (Some authors also write $f.(x + y)$ with a "dot")
- *They are used to define intervals, for example, $(1, \pi)$ may be shorthand for:* «the set of real numbers larger than 1 and smaller than π;» *in formula*

  $$(1, \pi) = \{t \in \mathbb{R} : 1 < t < \pi\} ;$$

  this extends to ordered sets, see Sect. §3.d.d.
- *They are used to represent elements of the Cartesian product; for example, $(1, \pi)$ is point in $\mathbb{R}^2$ with 1 as abscissa and π as ordinate.*

[†1] $\wedge$ and $\vee$ are also used in partially ordered set, but we will not discuss their definition in this text.

[†2] Or, more in general, if x, y are elements of a *ring* where multiplication is denoted by justaposition of symbols.

[†3] Again, more in general, if f, x, y are elements of a *ring* where multiplication is denoted by justaposition.

While the first and second situations are usually discernable and recognizable, the third and fourth can cause confusion.

Some care is needed in parsing statements involving Cartesian products of ordered sets, such as: «a point (x, y) in the rectangle R of the plane that is the product $R = (0, 1) \times (2, 4)$». *Here (x, y) is a point in $\mathbb{R}^2$ whereas $(0, 1), (2, 4)$ are intervals in $\mathbb{R}$.*

To avoid confusion, we may use a different notation for points and/or for intervals: many symbols that are similar to "parentheses" are available nowadays in the extended Unicode codespace, and are available to LaTeX users through the `unicode-math` *package.*

For example, in the above statement, we may use this (non-standard) notation: use barred parentheses $⦇\ldots⦈$ to denote the point in $\mathbb{R}^2$ with x as abscissa and y as ordinate; use double parentheses $⦅a, b⦆ = \{t \in \mathbb{R} : a < t < b\}$ for intervals; so as to obtain «a point $⦇x, y⦈$ in the rectangle R of the plane that is the product $R = ⦅0, 1⦆ \times ⦅2, 4⦆$». *In this case, for typographic consistency, we may use at the same time double brackets for closed-ended intervals, such as $⟦2, 4⟧$.*

This may be considered overkill for this example. But the situation can be more complicated, though!

For example, we may be dealing with intervals of elements of an ordered set X, that is also a Cartesian product $X = X_1 \times X_2$ of ordered sets X_1, X_2 (!) [†4] *In that case, we should first label the orders, for example: $\le_1$ being the order relation on X_1, $\le_2$ being the order relation on X_2, and $\preceq$ being the order relation on X; and use a (non-standard) notation for intervals, such as*

$$⦅a, b⦆_1 = \{t \in X_1 : a <_1 t <_1 b\}$$

for open-ended intervals in the first set (with extremes $a, b \in X_1$),

$$⦅z, w⦆_{\preceq} = \{x \in X : w \prec x \prec z\}$$

for open-ended intervals in the Cartesian product X (with extremes $z, w \in X$), and so on. Again, for typographic consistency, we may use double brackets for closed-ended intervals, such as

$$⟦a, b⟧_1 = \{x \in X_1 : a \le_1 x \le_1 b\}$$

and so on.

In the following we will often use the usual parentheses $()$; but in certain contexts we will use the notation proposed in this note (when it could help in understanding the text).

See also Remarks 3.a.6 and 6.1.

[†4] BTW, there is a standard method to order a Cartesian product of ordered sets, see Sect. §3.d.b.

§3 Fundamentals [00C]

§3.a Logic [1YS]

In the next sections we will give some definitions; these are simplified, but sufficient to cope with the exercises. Readers interested in an in-depth study can consult a book on Logic such as [12]. [23H]

§3.a.a Propositions

Definition 3.a.1. *A logical proposition φ is an assertion that takes on the value of truth or falsehood.* [1VW] *(Solved on 2020-10-22)*

Example 3.a.2. *Examples:* [1VX]

- *”the snow is white”,*
- *”the Earth has a diameter of about 12000km”,*
- *”a kg of bread costs 3$”.*

(One could argue philosophically about what is meant by ”truth”: in many areas the truth of a proposition is subjective, it can depend on the context, the interpretation, who does and who answers the question, on when the question is asked, etc etc; in mathematics the situation is simpler). [23J]

A proposition may depend on some variables. Examples:

- ”the person x by trade is a baker”,
- ”the number x is greater than 9”.

We write

$$P(x) \doteq \text{“the number x is larger than 9'}$$

to say that $P(x)$ is the symbol that summarizes the proposition written on the right.

Remark 3.a.3. *For the proposition to make sense, we will have to narrow down the scope of the variable to an appropriate set; in the first case, the set of human beings; in the second case, a numerical set (e.g. integers).* [23K]

At this level of discussion, the concept of ”set” is intuitive; we will see later that there is an axiomatic theory of sets, almost universally used in Mathematics; however, even the intuitive concept of set is widely used (See remark 3.b.16).

§3.a.b Propositional logic

Definition 3.a.4. *A **propositional logic** is a language, with associated an alphabet of variables (which for convenience in the following we will identify with the Italian alphabet) and a family of connectives* [†5] [00D]

negation, NOT	$\neg$
conjunction, AND	$\wedge$
disjunction, OR	$\vee$
implication	$\Rightarrow$
biconditional, iff	$\Leftrightarrow$

[†5] In logic texts, the symbol $\to$ is often used for the implication and the symbol $\leftrightarrow$ for the double implication

to these symbols we add parentheses, which are used to group parts of the formula (when there is a risk of ambiguity); the parentheses are omitted when the precedence of the operators allows; the operators are listed in the previous list in descending order of precedence. [†6]

Definition 3.a.5. ***Well-formed formulas*** *are* [00G] (Solved on 2022-10-11)

- *atomic formulas,* i.e. *composed of a single variable, or*
- *a formula of the type* $\neg(\alpha)$ *where* α *is a well-formed formula, or*
- – *a formula of the type* $(\alpha) \Rightarrow (\beta)$ *, or*
  – *a formula of the type* $(\alpha) \Leftrightarrow (\beta)$ *, or*
  – *a formula of the type* $(\alpha) \vee (\beta)$ *, or*
  – *a formula of the type* $(\alpha) \wedge (\beta)$ *,*

  where α, β *are two well-formed formulas.*

You can determine if a formula is well formed by making a finite number of checks using the previous rules: in fact the rules establish that any well-formed formula must be decomposable in terms of well-formed formulas that are shorter than it. So the statement *"this formula is well formed"* is "decidable". [†7] [1YK]

Remark 3.a.6. *In the definition 3.a.5 we speak of* atomic formulas, *i.e.* composed of a single variable*; we want to reflect on this. In programming languages we may use names composed of several letters to identify objects (variables, functions, etc.): such as* [228]

```
foo = 3 ;
bar = 7;
foo = foo + bar;
```

In mathematics this is unusual, since in a formula such as

$$xyz + abc$$

it would be difficult to understand if $\mathfrak{xyz}$ *is a variable, or the product of three variables* x, y, z*. For this reason, usually, in mathematics the identifiers are composed of a single letter; some notable functions are an exception, such as* sin, cos, exp, log*...etc. However, this creates some problems when you want to express a formula where there are many variables; for this reason, letters from the Greek alphabet are also used, and even Hebrew, in particular "aleph"* $\aleph$ *and "beth"* $\beth$*; and the letters are also accompanied by indexes, subscript as* x_1, x_2, x_3 *or superscript* x^1, x^2, x^3 *(being careful not to be confused with the exponentiation); then there are variants expressed with the signs* $\hat{x}, \overline{x}, \tilde{x}, x'$ *(being careful not to get confused with derivatives); and there are choices of fonts, such as* "calligraphic" $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots$*, the* "fraktur" $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \ldots \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ *or the* blackboard bold $\mathbb{a}, \mathbb{b}, \mathbb{c}, \mathbb{d} \ldots \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$.

Definition 3.a.7. *An* ***evaluation*** *assigns to each variable a value of "true" or "false".* [00J]

[†6] Some scholars use a different order of precedence, some consider "the implication" as preceding the "disjunction". For this reason it is always better to use parentheses to group the parts of phrase where these connectives are used.

[†7] The precise definition of "decidable" goes beyond these notes. Think of an algorithm written on the computer that, given a formula, with a finite number of computations answer "well formed" or "not well formed". Note, however, that the number of checks to be done grows exponentially with the length of the formula.

Knowing the value of the variables, and using the known truth tables for connectives [†8]*, we can calculate the value of each well-formed formula.*

So a well-formed formula is a "*logical proposition*" as it takes on the value of truth or falsehood, depending on the value given to its free variables. We can broaden the definition by adding that the propositions seen in the previous section they are "atomic formulas"; For example,

$$\text{"}x \text{ is a number less than 3"} \wedge \text{"}y \text{ is an even number"}$$

it will also be a "well-formed formula".

For convenience, in this Section, we also add to the language the constants V and F which are respectively always true and always false, in every evaluation. [†9] In the construction of well-formed formulas they are treated as variables. Note that we have not introduced the equality connective "=". When all variables can only take true/false values, the equality $a = b$ can be interpreted as $a \iff b$. In more general contexts (as in the case of set theory) instead, "equality" needs a precise definition.

Exercises

E3.a.8 Complete the following truth table [1VY]

P	Q	¬ P	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
V	V						
V	F						
F	V						
F	F						

Hidden solution: [UNACCESSIBLE UUID '1VZ']

E3.a.9 Tell which formulas are well formed, and add parentheses to highlight the order of precedence. [0OK]

$$a \wedge \neg b \wedge c \wedge d$$
$$\neg a \vee b \wedge c \Rightarrow d$$
$$a \Rightarrow \neg b \wedge c \vee d$$
$$a \wedge b \vee c \Leftrightarrow \neg c \Rightarrow d$$
$$a \vee b \neg c \vee d$$

Hidden solution: [UNACCESSIBLE UUID '0OM']

E3.a.10 A well-formed formula in propositional logic is a **tautology** if for each *evaluation* the formula is always true. Suppose A, B, C are well-formed formulas. Show that the following properties of connectives are tautologies. [†10] [0ON]

[†8] See 3.a.8
[†9] We can get rid of constants V and F by defining them as $V = A \vee \neg A$ and $F = \neg V$.
[†10] These lists are taken from Section 1.3 in [12], or [29].

$A \Rightarrow A$	identity law	
$\neg(\neg A) \Leftrightarrow A$	law of double negation	
$A \vee A \Leftrightarrow A \ , \ A \wedge A \Leftrightarrow A$	laws of idempotence	
$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$	law of opposition, or of the contrapositive [†11]	(3.a.11)
$(A \Rightarrow B) \Leftrightarrow (\neg A \vee B) \Leftrightarrow (\neg(A \wedge \neg B))$	equivalence of implication, conjunction and disjunction	(3.a.12)
$A \wedge B \Leftrightarrow \neg(\neg A \vee \neg B)$	first law of De Morgan	(3.a.13)
$A \vee B \Leftrightarrow \neg(\neg A \wedge \neg B)$	second law of De Morgan	(3.a.14)
$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$	distributive property of the conjunction with respect to the disjunction	(3.a.15)
$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	distributive property of the disjunction with respect to the conjunction	(3.a.16)
$A \wedge B \Leftrightarrow B \wedge A$	commutative property of $\bigwedge$	
$A \vee B \Leftrightarrow B \vee A$	commutative property of $\bigvee$	
$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$	associative property of $\bigwedge$	
$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$	associative property of$\bigvee$	(3.a.17)

These last two properties allow to omit parentheses in sequences of conjunctions or disjunctions.

The property (3.a.12),(3.a.13),(3.a.14) they say that we could base all logic on connectives alone $\neg$ and $\wedge$, (or on $\neg$, $\vee$).

[†11] The clause $(\neg B \Rightarrow \neg A)$ is called *"contrapositive"* of $(A \Rightarrow B)$.

Other important tautologies, often used in logical reasoning.

$A \vee \neg A$	excluded middle	
$\neg(A \wedge \neg A)$	law of non-contradiction	
$(A \wedge (A \Rightarrow B)) \Rightarrow B$	modus ponens	(3.a.18)
$(\neg B \wedge (A \Rightarrow B)) \Rightarrow \neg A$	modus tollens	(3.a.19)
$\neg A \Rightarrow (A \Rightarrow B)$	negation of the antecedent	
$B \Rightarrow (A \Rightarrow B)$	affirmation of the consequent	
$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)$	exporting	
$((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$	proof by parts	
$((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$	proof by cases	
$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$	hypothetical syllogism, or transitivity of implication	
$(A \vee (A \wedge B)) \Leftrightarrow A \wedge (A \vee B) \Leftrightarrow$		
$(A \vee F) \Leftrightarrow (A \wedge V) \Leftrightarrow A$	absorption laws	
$F \Rightarrow B$	first law of Pseudo Scotus, or *ex falso sequitur quodlibet*	
$A \Rightarrow (\neg A \Rightarrow B)$	second law of Pseudo Scoto	
$(\neg A \Rightarrow F) \Leftrightarrow A$	proof by contradiction	
$((A \wedge \neg B) \Rightarrow F) \Leftrightarrow (A \Rightarrow B)$	proof by contradiction, with hypothesis and thesis	
$(\neg A \Rightarrow A) \Rightarrow A$	consequentia mirabilis	(3.a.20)

E3.a.21 Show the validity of the following tautology [22C]

$$((\neg A \wedge B) \Rightarrow C) \iff ((\neg C \wedge B) \Rightarrow A)$$

Then use the 3.k.6 exercise to turn it into a Venn diagram with three sets. *Hidden solution:* *[UNACCESSIBLE UUID '22D']*

E3.a.22 Show that the implication connective $\Rightarrow$ is neither commutative nor associative. [†12] *Hidden solution:* *[UNACCESSIBLE UUID '2G9']* [2G8]

§3.a.c First-order logic

In the first order logic we add the connectives $\forall$, which reads *"for each"* and $\exists$, which reads *"exists"*. We must therefore enlarge the family of **well-formed formulas**.

Definition 3.a.23. *A formula is well formed if it meets all the rules in the list in 3.a.5 and this additional rule:* "given a well-formed formula ϕ where the variable x is free, a formula of the form "$\forall x, \phi$", or "$\exists x, \phi$" is a well-formed formula." [00Q]

*We will say that a variable x is **free** in a well-formed formula if*

- *the formula is atomic and the variable x appears in it; or if*
- *the formula is of the form $\neg\alpha$ and the variable x is free in α; or even if*
- *the formula is of the form $\alpha \wedge \beta, \alpha \vee \beta, \alpha \Rightarrow \beta, \alpha \iff \beta$ (or other logical connective introduced later) and the variable x is free in α or β.*

[†12] This exercise came about during a discussion with Anton Mennucci.

So in the formulas $(\forall x, \phi)$ *or* $(\exists x, \phi)$*, the variable* x *is no longer free; we will say that* "the variable is quantified".

In every part of a formula where a variable is quantified this variable can be replaced with every other variable.

Remark 3.a.24. *The variable* x *, which is quantified in a part of a formula, is again free if it is reused in another piece of the formula; this is syntactically permissible but makes the formula less readable, as in this example that uses the language of set theory* [1X1] *(Solved on 2022-10-11)*

$$A \subseteq \mathbb{N} \land x \in \mathbb{N} \land x \geq 4 \land (\forall x \in A, x \leq 10)$$

which should be written as

$$A \subseteq \mathbb{N} \land x \in \mathbb{N} \land x \geq 4 \land (\forall y \in A, y \leq 10)$$

renaming the variable inside the part where it is quantified.

Remark 3.a.25. *It is assumed as an axiom that* [2DC]

$$\neg(\forall x, \phi) \Leftrightarrow (\exists x, \neg\phi) \quad . \tag{3.a.26}$$

(In Sec. 2.1 in [12] indeed $(\forall x, \phi)$ *is presented as short form for* $\neg(\exists x, \neg\phi)$*).*

Note that, in many examples, quantified variables are assumed to be elements of a "set".

Definition 3.a.27. *Given two variables* x, y *we will write* $x \in y$ *to say that* "x is an element of the set y"*. Equivalent expressions are* "x is a member of y", "x belongs to y" *or just simply* "x is in y". [1X2]

The formula $(x \in y)$ *is equivalent to* $(y \ni x)$*; the negations are* $(x \notin y) \doteq \neg(x \in y)$ *and* $(y \not\ni x) \doteq \neg(y \ni x)$*.*

The formula $(x \in y)$ *(as all other variants) takes value of truth/falsehood and therefore can be used as* atom *in the construction of a well-formed formula.*

Definition 3.a.28. *We usually write* [0OR]

"$\forall x \in A, P(x)$" *to say* "for every x in A $P(x)$ holds",
or
"$\exists x \in A, P(x)$" *to say* "there is a x in A for which $P(x)$" *holds;*

(where A *is a set); to link these writings to the previous definitions, we decide that the previous writings are abbreviations for*

$$\forall x \in A, P(x) \doteq \forall x, x \in A \Rightarrow P(x) \quad ,$$
$$\exists x \in A, P(x) \doteq \exists x, x \in A \land P(x) \quad .$$

Note that these RHS are "well-formed formulas". See also the exercise 3.a.35.

We use the term "together" informally here, see footnote 3.b.16.

Remark 3.a.29. *Note that "*$\forall x \in A, \varphi$*" is true if* A *is the empty set; this is consistent with what was discussed in the exercise 3.a.35. This has though a striking consequence: the implication* [0OS] *(Solved on 2022-10-11)*

$$(\forall x \in A, \varphi) \Rightarrow (\exists x \in A, \varphi)$$

is always valid when A *is a non-empty set, but is instead false when* $A = \emptyset$*.*

Since an element of a set may not have a truth/falsehood value, we enrich the language by adding the "*logical propositions*".

Definition 3.a.30. *A **logical proposition** ϕ is an assertion that assumes value of truth or falsehood depending on the value given to the its free variables, and only from that.* [00T] *(Solved on 2021-10-18)*

An example of a logical proposition would be: *"n is an even number"*. We can use logical propositions as atoms in the construction of well-formed formulas.

Exercises

E3.a.31 Let X, Y sets. Let ϕ, ψ logical propositions be; x, a are free variables in ϕ, and y, b are free in ψ. We also assume that a, b can only be true or false, while $x \in X, y \in Y$. Consider the following formulae. Which ones are well formed? What variables are free in them? [00V]

$$b \wedge (\forall x, \phi)$$
$$(\exists y, \psi) \vee (\forall x, \phi)$$
$$\forall x, \forall b, \big(\phi \wedge (\psi \vee b)\big)$$
$$a \vee (\forall x, \forall a, \phi)$$
$$(\exists x, \psi) \wedge (\forall x, \phi)$$

Hidden solution: *[UNACCESSIBLE UUID '00W']*

E3.a.32 Consider a proposition $P(u, \ell)$ dependent on two free variables u (which takes values in the set of people), and ℓ (in the set all the jobs), and which is worded as follows: *'Person u knows how to do the job ℓ'*. [00X]

Express the following formulas in English

$$\exists u \exists \ell P(u, \ell) \ , \forall u \exists \ell P(u, \ell) \ , \exists \ell \forall u P(u, \ell) \ ,$$
$$\forall \ell \exists u P(u, \ell) \ , \exists u \forall \ell P(u, \ell) \ , \forall u \forall \ell P(u, \ell) \ .$$

Hidden solution: *[UNACCESSIBLE UUID '00Y']*

E3.a.33 What implications are there among the previous formulas? [00Z] *(Proposed on 2021-10-21)*

Hidden solution: *[UNACCESSIBLE UUID '010']*

E3.a.34 You may prove that [011]

$$\Big((\forall x, \varphi(x)) \wedge (\forall y, \psi(y))\Big) \iff \Big(\forall x\, (\varphi(x) \wedge \psi(x))\Big)$$

E3.a.35 As already commented in 3.a.28, given A a set, and $P(x)$ a logical proposition dependent from a free variable x, we usally write [016]

$$\forall x \in A, P(x) \quad , \quad \exists x \in A, P(x)$$

however

$$\forall x \in A, P(x) \text{ summarizes } \forall x, (x \in A) \Rightarrow P(x) \ ,$$
$$\exists x \in A, P(x) \text{ summarizes } \exists x, (x \in A) \wedge P(x) \ ;$$

where the "extended" versions are well-formed formulas.

Using this extended version you can prove that the two propositions

$$\neg(\forall x \in A, P(x)) \ , \ \exists x \in A, (\neg P(x)) \ .$$

are equivalent, in the sense that from one it is possible to prove the other (and vice versa). In the proof use only tautologies (listed in 3.a.10) and in particular the equivalence of the formula "$P \Rightarrow Q$" with "$(\neg P) \vee Q$" [†13] , and finally the equivalence between "$\neg\exists x, Q$" and "$\forall x, \neg Q$" [†14].

Replacing $P(x)$ with $\neg P(x)$ and using the tautology of double negation finally results in

$$\forall x \in A, (\neg P(x)) \ , \ \neg(\exists x \in A, P(x))$$

are equivalent.

Hidden solution: [UNACCESSIBLE UUID '017']

E3.a.36 Given A a set, and $P(x)$ a proposition dependent on a free variable x, we usually write [013]

$$\exists! x \in A, P(x)$$

when there is one and only one element x of A for which $P(x)$ is true. Define this notation with a well-formed formula. (Note that you will need to use the equality connective, because you must be able to express the idea of "unique", which needs of a method to be able to tell when two objects are distinguishable and when they are not).

Hidden solution: [UNACCESSIBLE UUID '015']

§3.b Set theory [1YT]

§3.b.a Naive set theory [242]

As already explained in Definition 3.a.27, in set theory, the connective "∈" is added; given two sets z, y the formula $x \in y$ reads *"x belongs to y"* or more simply *"x is in y"*, and indicates that x is an element of y.

It is customary to indicate the sets using capitalized letters as variables.

Definition 3.b.1. *We also add the connective $a = b$ between sets, which is true when* [1Y8] *(Solved on 2022-10)*

$$\forall x, x \in a \iff x \in b \ .$$

*This is the **axiom of extensionality**.*

This says that two sets a and b are equal when they have the same elements; that is, it excludes that a set can have some other property that distinguishes it [†15]. [226]

Definition 3.b.2. *For convenience, the $a \subseteq b$ connective is used to indicate that a is a subset of b; formally this is defined by* [227]

$$\forall x, x \in a \Rightarrow x \in b \ .$$

$b \supseteq a$ is equivalent to $a \subseteq b$.

Obviously $a = b \iff ((a \subseteq b) \wedge (b \subseteq a))$. Note that $a \subseteq a$.

[†13] Tautology in eqn. (3.a.12).

[†14] Already discussed in eqn.(3.a.26).

[†15] One could imagine a set theory in which the parentheses can be "red" or "blue", and the equality between sets occur when the elements and colors are the same. In the usual theory the parentheses are always black.

It is usual to write $x \notin y$ for $\neg(x \in y)$, $x \not\subseteq y$ for $\neg(x \subseteq y)$ and so on.

Remark 3.b.3. *There are also other symbols used. Some texts use $a \subset b$ to indicate that $a \subseteq b$ but $a \neq b$ (as in the notes [2]); others use a more expressive writing such as $a \subsetneq b$ to say that $a \subseteq b$ but $a \neq b$. (Some even use $a \subset b$ instead of $a \subseteq b$, unfortunately — e.g. [13]).* [1W0]

We also define the constant $\emptyset$, also referred to as $\{\}$, which is the empty set,[†16] that is uniquely identified by the property

$$\forall x, \neg x \in \emptyset \quad .$$

Some fundamental concepts are therefore introduced: union, intersection, symmetric difference, power set, Cartesian product, relations, functions *etc*.

Definition 3.b.4. *Given I a non-empty family of indices and given C_i sets (one for each $i \in I$), then the* ***union*** [1Y2]

$$\bigcup_{i \in I} C_i$$

is a set, which contains all (and only) the elements of all sets C_i; in formula[†17]

$$\bigcup_{i \in I} C_i \stackrel{def}{=} \{x : \exists i \in I, x \in C_i\} \quad .$$

If only two sets are given C_1, C_2, we usually write $C_1 \cup C_2$ to indicate the union; and similarly when finite sets are given.

Definition 3.b.5. *Given I a non-empty family of indexes and given C_i sets (one for each $i \in I$), we define the* ***intersection*** [1W1]

$$\bigcap_{i \in I} C_i$$

which is the set that contains the elements that belong to all sets C_i (for all $i \in I$).

If only two sets are given C_1, C_2, we usually write $C_1 \cap C_2$ to indicate the intersection, and you have

$$C_1 \cap C_2 \stackrel{def}{=} \{x \in C_1 \cup C_2 : x \in C_1 \wedge x \in C_2\} \quad ;$$

and similarly when finite sets are given.

The power set is defined as in ZF:5.

Definition 3.b.6. *Other operators between sets are:* [23S]

- *the difference*

$$A \setminus B \stackrel{def}{=} \{x \in A : x \notin B\} \quad ;$$

- *if the set A is clearly specified by the context, and if $B \subseteq A$, it is common to write $B^c \stackrel{def}{=} A \setminus B$; B^c is said to be the complement of B in A;*
- *the symmetric difference*

$$A \Delta B \stackrel{def}{=} (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = \{x \in A \cup B : x \in A \iff x \notin B\} \quad ;$$

where A, B are sets.

[†16] In Zermelo–Fraenkel axiomatic theory, the existence of $\emptyset$ is an axiom.
[†17] This is a more manageable version of the official axiom. The official definition is located in ZF:4.

Exercises

E3.b.7 Prove that $A = B$ if and only if $((A \subseteq B) \wedge (B \subseteq A))$. *Hidden solution:* [UNACCESSIBLE UUID '1W7'] [1W6] (Proposed on 2021-10-18)

E3.b.8 Represent operations [1W8]

- $\cup$ union
- $\cap$ intersection
- $\setminus$ difference
- Δ symmetrical difference

between two sets using Venn diagrams.

E3.b.9 Prerequisites:3.b.8.Use the above Venn diagrams to show that in general $(A \cup B) \cap C \neq A \cup (B \cap C)$. [1W9]

E3.b.10 Show that if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$ *Hidden solution:* [UNACCESSIBLE UUID '1WD'] [1WB]

E3.b.11 Explain why the union operation $A \cup B$ between two sets is commutative, and show that it is associative; similarly for the intersection; finally show that the union distributes over the intersection, and also that the intersection distributes over the union. *Hidden solution:* [UNACCESSIBLE UUID '1W3'] [1W2]

E3.b.12 Consider the sets: [1WF] (Proposed on 2022-12)

- P all the professors,
- S all scientists,
- F the set of philosophers,
- M the set of mathematicians.

For each of the following sentences, write a formula that represents it, using the above sets, the empty set, relations $\subseteq, =, \neq$, and set operations $\cup, \cap, \setminus$.

- not all professors are scientists
- some mathematician is philosopher;
- if a philosopher is not a mathematician then s/he is a professor;
- all philosophers are scientists or professors, but not mathematicians;
- if there is a mathematician who is also a scientist, then s/he is neither a philosopher nor a professor.

Hidden solution: [UNACCESSIBLE UUID '1WG']

E3.b.13 Let U be the set of human beings, A the set of animals and M the set of mortal creatures; convert the following syllogism into formulas and prove it: [1Y4]

every man is an animal, every animal is mortal, therefore every man is mortal.

E3.b.14 Explain the formula $\bigcup_{B\in\mathcal{P}(A)} B$ using the definition 3.b.4 of the axiom of union. Then show that $A = \bigcup_{B\in\mathcal{P}(A)} B$. [1WC]

See also 3.b.25 where the same result is obtained starting from the axiom of union as defined in ZF:4 in the Zermelo–Fraenkel axiomatic.

Hidden solution: [UNACCESSIBLE UUID '1WV']

E3.b.15 Let I, C_i as in 3.b.4 and let A be a set; prove that [24P]

$$\bigcup_{i\in I} C_i \subseteq A$$

if and only if

$$\forall i \in I,\ C_i \subseteq A\ .$$

Remark 3.b.16. *A distinction is made between an* informal set theory *and a* formal set theory. [†18] [01J]

Informal set theory exploits all notions previously listed, but does not investigate the fundamentals, that is, the axiomatization. For this approach we recommend the text [9]; or [28] for a brief discussion.

The most widely used formal set theory *is the Zermelo–Fraenkel axiomatic, that we will shortly recall in next Section. See Chap. 6 in [12] (for a brief introduction [30] can also be fine).*

In Zermelo—Fraenkel's axiomatic set theory, all variables represent sets, so variables do not have a meaning of truth or falsehood. For this reason, in the definitions 3.a.5 and 3.a.23 of well-formed formula *changes the concept of "atom". A An atom is now a formula of the form $a \in b$ that has truth/falsehood value.*

While in formal theory all the elements of language are sets, in practice we tend to distinguish between the sets, and other objects of Mathematics (numbers, functions, etc etc); for this in the following we will generally use capital letters to indicate the sets, and lowercase letters to indicate other objects.

§3.b.b Zermelo–Fraenkel axioms [241]

We now briefly discuss the axioms of Zermelo–Fraenkel set theory.

ZF:1 Axiom of extensionality, already seen above in 3.b.1.

ZF:2 The empty set $\emptyset$ is a set. The formula for this axiom is [014]

$$\exists X : \forall Y \neg (Y \in X)$$

and by the preceding axiom, X is unique, so it is denoted by $\emptyset$.

ZF:3 **Axiom of pairing**. Given any two sets X and Y there exists a set Z, denoted by $Z = \{X, Y\}$, whose only two elements are X and Y. In formula [1Y3]

$$\forall X, Y \exists Z : \forall W (W \in Z) \iff (W = X) \vee (W = Y) \quad .$$

Again, by the axiom of extensionality 3.b.1, the set Z unique.

ZF:4 The **axiom of union**[†19] says that for each set A there is a set B that contains all the elements of the elements of A; in symbols, [026]

$$\forall A \exists B, \forall x, (x \in B \iff (\exists y, y \in A \wedge x \in y)) \quad .$$

This implies that this set is unique, by the axiom of extensionality 3.b.1; we indicate this set B with $\underline{\bigcup} A$ (so as not to confuse it with the symbol already introduced before).

For example if

$$A = \{\{1, 3, \{5, 2\}\}, \{7, 19\}\}$$

then

$$\underline{\bigcup} A = \{1, 3, \{5, 2\}, 7, 19\} \quad .$$

Given $A_1, \dots A_k$ sets, let $D = \{A_1, \dots A_k\}$ [†20] we define

$$A_1 \cup A_2 \dots \cup A_k \stackrel{\text{def}}{=} \underline{\bigcup} D \quad .$$

ZF:5 The **axiom of the power set** says that for every set A, there is a set $\mathcal{P}(A)$ whose elements are all and only subsets of A. A shortened definition formula is [1Y1]

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{B : B \subseteq A\} \quad .$$

$\mathcal{P}(A)$ is also called *set of parts*.

In the formal language of the Zermelo-Fraenkel axioms, the axiom is written:

$$\forall A, \exists Z, \forall y, y \in Z \iff (\forall z, z \in y \implies z \in A) \quad ;$$

this formula implies that the power set Z is unique, therefore we can denote it with the symbol $\mathcal{P}(A)$ without fear of misunderstandings.

Note that

$$(\forall z, z \in y \implies z \in A)$$

can be shortened with $y \subseteq A$ and therefore the axiom can be written as

$$\forall A, \exists Z, \forall y, y \in Z \iff (y \subseteq A) \quad ;$$

then using the extensionality, we obtain that

$$Z = \{y : (y \subseteq A)\} \quad .$$

ZF:6 Axiom of infinity (see 3.h.11)

ZF:7 The **axiom of specification**, which reads [1Y0]

If A is a set, and $P(x)$ is a logical proposition, then $\{x \in A : P(x)\}$ is a set.

[†18] See the introduction to Chap. 6 in [12] for a discussion comparing these two approaches.
[†19] This is the "official" version of Zermelo–Fraenkel. However, the simplified version 3.b.4 is often used
[†20] The existence of this set can be proven, see 3.b.35

Formally, setting $B = \{x \in A : P(x)\}$,

$$\forall X, X \in B \iff X \in A \wedge P(x) \quad .$$

This axiom avoids Russell's paradox: let A be the set of x such that $x \notin x$, then you have neither $A \in A$ nor $A \notin A$.

ZF:8 Axiom of good foundation, or regularity (see 3.b.36)

ZF:9 Axiom of replacement

(We have omitted the definitions of *"Axiom of replacement"*; you can find it in Chap.1 Sec.16 in [2] or Chap. 1 in [13]).

A further axiom is the *Axiom of Choice*; it will be discussed in Sec. §3.b.c.

Remark 3.b.17. *Zermelo–Fraenkel set theory with the axiom of choice included is abbreviated ZFC, whereas ZF refers to the axioms of Zermelo–Fraenkel set theory (without the axiom of choice).* [2DX]

Remark 3.b.18. *This wording is commonly used:* "let I be a non-empty set of indices, and A_i a family of setsCDLeng indexed by $i \in I$"; *this, in axiomatic theory, should be written as* "let I be a non-empty set, let X be a set, and $A : I \to \mathcal{P}(X)$ a function; we will write A_i instead of $A(i)$". *(With this writing we have that A_i are all subsets of X).* [01M]

Exercises

E3.b.19 The notation in ZF:4 differs from the usual one, which is $\bigcup_{i \in I} C_i$, where I is a non-empty family of indices and C_i are sets; as seen in 3.b.4. [23W]

How can you define $\bigcup_{i \in I} C_i$ using the axiom of union presented ZF:4? *(Sugg. re-read the note 3.b.18)*

Eventually you should obtain

$$\forall x, x \in \bigcup_{i \in I} C_i \iff \exists i \in I, x \in C_i \quad . \tag{3.b.20}$$

Hidden solution: *[UNACCESSIBLE UUID '027']*

E3.b.21 Prove that the definition 3.b.5 of intersection is well posed, using the Z-F axioms. Eventually prove also that [23T] *(Proposed on 2022-10-11)* *(Solved on 2022-10-25)*

$$\forall x, x \in \bigcap_{i \in I} C_i \iff (I \neq \emptyset \wedge \forall i \in I, x \in C_i) \quad . \tag{3.b.22}$$

Hidden solution: *[UNACCESSIBLE UUID '23V']*

E3.b.23 `Prerequisites:3.b.21,ZF:4,ZF:7,3.a.29.` Let A be a non-empty set; we define B as the set that contains all the elements that are in all the elements of A. Write a well-formed formula that defines B, prove that B is indeed a set, and show that it is unique; for symmetry with the axiom ZF:4 we will indicate it with [252]

$$B = \underline{\bigcap} A \quad .$$

It is related to the usual notation by the relation

$$\underline{\bigcap} A = \bigcap_{x\in A} x \quad .$$

Hidden solution: [UNACCESSIBLE UUID '254']

E3.b.24 Prerequisites:3.b.19,3.b.21. Now that you have correctly defined the union 3.b.5 and the intersection 3.b.4 using the Z-F axioms, tell what value are assumed by [247] (Solved on 2022-10-25)

$$\bigcap_{i\in I} C_i$$

and

$$\bigcup_{i\in I} C_i$$

when I is the empty set. *Hidden solution:* [UNACCESSIBLE UUID '249']

E3.b.25 Prerequisites:ZF:4. Using the definition of $\underline{\bigcup}$ presented in ZF:4, show that $A = \underline{\bigcup}(\mathcal{P}(A))$. [028]

E3.b.26 Given a set X and I, C_i as in 3.b.5 and 3.b.4, show that [248] (Solved on 2022-10-25)

$$X \setminus \left(\bigcap_{i\in I} C_i \right) = \bigcup_{i\in I} (X \setminus C_i) \quad . \tag{3.b.27}$$

What happens when I is the empty set?

Hidden solution: [UNACCESSIBLE UUID '24B']

E3.b.28 If A is a set of n elements ($n \geq 0$ natural number) then how many elements are there in $\mathcal{P}(A)$? [1W4] (Proposed on 2022-12)

E3.b.29 Write explicitly $\mathcal{PPP}(\emptyset)$. How many elements does it have? *Hidden solution:* [UNACCESSIBLE UUID '1WX'] [023]

E3.b.30 Let be given a, b, x, y. [1Y9]

1. Show that in the hypothesis

$$\{a, b\} = \{x, y\}$$

you have that

$$(a = b) \iff (x = y) \iff a = b = x = y \quad .$$

2. In particular, you deduce that if

$$\{a\} = \{x, y\}$$

then $a = x = y$.

3. Then show that if we assume that the four elements a, b, x, y are not all the same, then we have

$$\{a, b\} = \{x, y\}$$

if and only if $a = x \wedge b = y$ or $a = y \wedge b = x$.

To show the above be as precise as possible: use the axiom of extensionality 3.b.1, the axiom of pairing ZF:3 and the tautulogies shown in the previous section (or other elementary logical relationships). *Hidden solution:* [UNACCESSIBLE UUID '1YB']

E3.b.31 Prerequisites:3.b.30. The *ordered pair* is defined as [01N] (Solved on 2022-10-25)

$$(x, y) \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\} \quad ;$$

(note that the *axiom of pairing* ZF:3 guarantees us that this is a good definition); show that

$$(a, b) = (x, y) \iff (a = x \wedge b = y) \quad . \tag{3.b.32}$$

(First solution that doesn't use 3.b.30) Hidden solution: [UNACCESSIBLE UUID '1WZ'])

(Second solution using 3.b.30) Hidden solution: [UNACCESSIBLE UUID '1YC'])

E3.b.33 Prerequisites:3.b.30,3.b.36. Let's imagine a different definition for the *ordered pair*, defined as [1YD]

$$(\!(x, y)\!) \stackrel{\text{def}}{=} \{x, \{x, y\}\} \quad ;$$

show that

$$(\!(a, b)\!) = (\!(x, y)\!) \iff (a = x \wedge b = y) \quad . \tag{3.b.34}$$

To show it you will need 3.b.36. *Hidden solution:* [UNACCESSIBLE UUID '1YF']

E3.b.35 Show that, given $a_1, \dots a_k$ sets, there is a set that contains all and only these elements. This set is usually denoted by $\{a_1, \dots a_k\}$. [029] (Solved on 2022-10-25)

Hidden solution: [UNACCESSIBLE UUID '02B']

E3.b.36 The **axiom of good foundation** (also called **axiom of regularity**) of the Zermelo–Fraenkel theory says that every non-empty set X contains an element y that is disjoint from X; in formula [01R] (Solved on 2022-10-25)

$$\forall X, X \neq \emptyset \Rightarrow (\exists y\, (y \in X) \wedge (X \cap y = \emptyset))$$

(remember that every object in the theory is a set, so y is a set). Using this axiom prove these facts.

- There is no set x that is an element of itself, that is, for which $x \in x$.
- More generally there is no finite family $x_1, \dots x_n$ such that $x_1 \in x_2 \in \dots \in x_n \in x_1$.
- There is also no $x_1, \dots x_n, \dots$ sequence of sets for which $x_1 \ni x_2 \ni x_3 \ni x_4 \dots$.

Hidden solution: [UNACCESSIBLE UUID '01S']

E3.b.37 Prerequisites:3.b.36. Show that for every x there is a y such that $y \notin x$ *Hidden solution:* [UNACCESSIBLE UUID '01X'] [01W] (Solved on 2021-10-21)

E3.b.38 Show instead that the **axiom of infinity**, and the consequent construction of the natural numbers seen in Sec. §3.h, implies that there is a sequence $x_1, \dots x_n, \dots$ of sets for which $x_1 \in x_2 \in x_3 \dots$. *Hidden solution:* [UNACCESSIBLE UUID '01Z'] [01Y]

E3.b.39 Prerequisites:3.b.41. Given A non-empty set show that there is a bijection $f : A \to B$ between A and a set B disjoint from A. [020] *(Solved on 2022-10-25)*

More generally, let I a non-empty set of indexes, and A_i a family of non-empty sets indexed by $i \in I$; [†21] show that there are bijections $f_i : A_i \to B_i$, where the sets B_i enjoy $\forall i \in I, \forall j \in I, B_i \cap A_j = \emptyset$ and for $j \neq j$ also $B_i \cap B_j = \emptyset$.

Hidden solution: [UNACCESSIBLE UUID '021']

E3.b.40 Prerequisites:ZF:5. [022]

Show that $X \subseteq Y$ if and only if $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. *Hidden solution:* [UNACCESSIBLE UUID '1WW']

E3.b.41 Using the definition of pair (a, b) as $\{\{a\}, \{a, b\}\}$ show that, given two sets x, y , for each $a \in x, b \in y$ you have [024]

$$(a, b) \in \mathcal{P}\mathcal{P}(x \cup y) \ .$$

Use this fact and the axiom of separation to justify axiomatically the definition of the **Cartesian product** $x \times y$.

Hidden solution: [UNACCESSIBLE UUID '025']

Remark 3.b.42. *In the exercise 3.b.35 the elements are identified using variables $a_1, \dots a_k$ that we may have denoted using other letters such as $a, b, c, d, \dots$. If we instead think of $a_1, \dots a_k \dots$ as values of a function $a_i = a(i)$, $a : I \to X$ then the set $\{a_1, \dots a_k \dots\}$ always exists (for any choice of I) since it is the image of the function $\{a_1, \dots a_k \dots\} = \{x \in X : \exists i \in I, x = a_i\}$.* [27F]

§3.b.c Zorn Lemma, Axiom of Choice, Zermelo's Theorem [23R]

There are three fundamental statements in set theory, Zorn's Lemma, the Axiom of Choice, and Zermelo's Theorem. It is proven, within the Zermelo–Fraenkel axiomatics, that these are equivalent. See in Chap. 1 in [2] for an elementary presentation, based on the above defined theory. [†22]

The first exercise presents some fundamental equivalent ways to state the Axiom of Choice.

Exercises

E3.b.43 Prerequisites:3.b.18, ZF:4,3.b.39 . Let I be a non-empty set of indexes, let A_i a family of non-empty sets indexed by $i \in I$. Recall that, by definition, the Cartesian product $\prod_{i \in I} A_i$ is the set of functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for each $i \in I$. [02H]

Show that the following are equivalent formulations of the **axiom of choice**.

- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- Given a family A_i as above, such that the sets are not-empty and pairwise disjoint, there is a subset B of $\bigcup_{i \in I} A_i$ such that, for each $i \in I$, $B \cap A_i$ contains a single element.

[†21] *Cf.* 3.b.18

[†22] This theory can be found in many books on Logic, such as [12, 13, 10], but the statements and proofs use a language and mathematical tools that may be too advanced for the intended audience of this book.

- Let S be a set. Then there is a function $g : \mathcal{P}(S) \to S$ such that $g(A) \in A$ for each nonempty $A \in \mathcal{P}(S)$.

Hidden solution: [UNACCESSIBLE UUID '02J']

Remark 3.b.44. *Attention! Suppose as above that the sets A_i are not empty. This is formally written as $\forall i \in I, \exists x \in A_i$. Intuitively this brings us to say that the element x depends on i, and therefore that $x = x(i)$. This step, as intuitive as it is, is exactly the axiom of choice.* [02K]

Exercises

E3.b.45 Find a non-empty set of indexes I, and, for each $i \in I$, non-empty sets A_i, so that there does not exists a subset B of $\bigcup_{i\in I} A_i$ with the property that, for each $i \in I$, $B \cap A_i$ contains a single element. *Hidden solution:* [UNACCESSIBLE UUID '2GG'] [2GF]

E3.b.46 Prerequisites:3.e.20.Consider the Zermelo-Fraenkel set theory, and this statement: [2BZ]

Given any A, B non-empty sets such that there exists a surjective function $g : B \to A$, then there exists an injective function $f : A \to B$ such that $g \circ f = Id_A$.

Prove that this statement implies the Axiom of Choice. *Hidden solution:* [UNACCESSIBLE UUID '2C0']

E3.b.47 Let V be a real vector space. Let $B \subseteq V$ be a subset. A **finite linear combination** v of elements of B is equivalently defined as [02D]

- $v = \sum_{i=1}^n \ell_i b_i$ where $n = n(v) \in \mathbb{N}$, $\ell_1, \dots, \ell_n \in \mathbb{R}$ and $b_1, \dots, b_n$ are elements of B;
- $v = \sum_{b\in B} \lambda(b)b$ where $\lambda : B \to \mathbb{R}$ but also $\lambda(b) \neq 0$ only for a finite number of $b \in B$.

We call $\Lambda \subseteq \mathbb{R}^B$ the set of functions λ as above, which are non-null only for a finite number of arguments; Λ is a vector space: so the second definition is less intuitive but is easier to handle.

We will say that B **generates** (or, **spans**) V if every $v \in V$ is written as finite linear combination of elements of B.

We will say that the vectors of B are **linearly independent** if $0 = \sum_{b\in B} \lambda(b)b$ implies $\lambda \equiv 0$; or equivalently that, given $n \geq 1$, $\ell_1, \dots, \ell_n \in \mathbb{R}$ and $b_1, \dots, b_n \in B$ all different, the relation $\sum_{i=1}^n \ell_i b_i = 0$ implies $\forall i \leq n, \ell_i = 0$.

We will say that B is an **algebraic basis** (also known as **Hamel basis**) if both properties apply.

If B is a basis then the linear combination that generates v is unique (i.e. there is only one function $\lambda \in \Lambda$ such that $v = \sum_{b\in B} \lambda(b)b$).

Show that any vector space has an *algebraic basis*. Show more in general that for each $A, G \subseteq V$, with A family of linearly independent vectors and G generators, there is an *algebraic basis* B with $A \subseteq B \subseteq G$.

Hidden solution: [UNACCESSIBLE UUID '02G']

The proof in general requires Zorn's Lemma; indeed this statement is equivalent to the Axiom of Choice; this was proved by A. Blass in [6]; see also Part 1 §6 [21].

E3.b.48 `Difficulty:*`.[†23] Consider the following quotient of the family of all integer valued sequences [02M]

$$\mathbb{X} = \{a : \mathbb{N} \to \mathbb{N}\}/\sim$$

where we define $a \sim b$ iff $a_k = b_k$ eventually in k.

We define the ordering

$$a \preceq b \iff \exists n \text{ s.t. } \forall k \geq n, a_k \leq b_k$$

that is, $a \preceq b$ when $a_k \leq b_k$ eventually. This is a preorder and

$$a \sim b \iff (a \preceq b \wedge b \preceq a)$$

so it passes to the quotient were it becomes an ordering, see Prop. 3.g.3.

Let a^k be an increasing sequence of sequences, that is, $a^k \preceq a^{k+1}$; we readily see that it has an upper bound b, by defining

$$b_n = \sup_{h,k \leq n} a^k_h .$$

We can then apply the Zorn Lemma to assert that in the ordered set $(\mathbb{X}, \preceq)$ there exist maximal elements.

Given a, b we define

$$a \vee b = (a_n \vee b_n)_n$$

then it is easily verified that $a \preceq a \vee b$. So this a *direct* ordering, see 3.d.15.

We conclude that the ordered set $(\mathbb{X}, \preceq)$ has an unique maximum, by 3.d.23.

This is though false, since for any sequence a the sequence $(a_n + 1)_n$ is larger than that.

What is the mistake in the above reasoning? What do you conclude about $(\mathbb{X}, \preceq)$?

Many other exercise need Zorn's Lemma, Axiom of Choice, Zermelo's Theorem in their proof; to cite a few: 3.e.20, 3.j.5, 3.j.39, 3.j.40, 3.j.41, 3.j.44.

Remark 3.b.49. "The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?" — *Jerry Bona*[†24] [02C]

This is a joke[†25]*: although the three are all mathematically equivalent, many mathematicians find the axiom of choice to be intuitive, the well-ordering principle to be counterintuitive, and Zorn's lemma to be too complex for any intuition.*

[†23] Originally published in https://dida.sns.it/dida2/Members/mennucci/curiosa/
[†24] As cited in [15].
[†25] Paragraph quoted from [46].

§3.c Relations [1YV]

Definition 3.c.1. *A **relation** between elements of two sets A, B is defined as a subset $R \subseteq A \times B$ of the cartesian product. Typically, the infix notation aRb is used instead of writing $(a, b) \in R$.* [1WY]

Definition 3.c.2. *A relation R between elements of A is said to be:* [23X]

- ***reflexive*** *if xRx for any $x \in A$;*
- ***irreflexive*** *or* ***anti-reflexive*** *if $\neg xRx$ for any $x \in A$;*
- ***symmetric*** *if xRy implies yRx for any $x, y \in A$;*
- ***antisymmetric*** *if aRb and bRa imply $a = b$, for any $a, b \in A$;*
- ***trichotomous*** *if for all $x, y \in A$ one and exactly one of xRy, yRx and $x = y$ holds;*
- ***transitive*** *if xRy and yRz imply xRz, for any $x, y, z \in A$.*

A relation R between elements of A and elements of B is said to be:

- ***injective*** *(also called left-unique) if xRy and zRy imply $x = z$, for any $x, z \in A, y \in B$;*
- ***functional*** *(also called right-unique) if xRy and xRz imply $y = z$, for any $x \in A, y, z \in B$; such a binary relation is called a "partial function" (see also §3.e,3.e.24);*
- ***total*** *(also called "left-total") if for any $x \in A$ there is a $y \in B$ such that xRy;*
- ***surjective*** *(also called "right-total") if for any $y \in B$ there is a $x \in A$ such that xRy.*

Definition 3.c.3. *An **equivalence relation** is a relation between elements of A that enjoys the properties: reflective, symmetrical, transitive.* [1W5]

Equivalence relations are typically denoted by symbols "$\sim$" , "$\approx$" , "$\simeq$" , "$\cong$" , "$\approxeq$" etc.

Definition 3.c.4. *An **order relation** (or simply **order**) is a relation between elements of A that enjoys the properties: reflective, antisymmetrical, transitive.* [1Y5]

*An order relation is **total** if all elements are **comparable**, i.e. if for every $a, b \in A$ you have $aRb \lor bRa$.*

*(When an order relation is not total, it is said to be **partial**).*

Symbols such as "$\le$" or "$\subseteq$" or "$\preceq$" or similar are generally used.

Remark 3.c.5. *The above is the definition in [2]; in other texts, a relation between elements of A that enjoys the properties: reflexive, antisymmetrical, transitive is straightforwardly called **partial order**.* (cf *Example 2.1.1 in [12] where moreover a* total order *is called* linear order*). For this reason we will sometimes add a "(partial)" to state that the order being discussed may be partial.* [24W]

Order relations are discussed in Section §3.d

Remark 3.c.6. *It is customary to write $a \geq b$ as a synonym for $b \leq a$. If $a \leq b \wedge a \neq b$ we will write $a < b$; similarly if $a \geq b \wedge a \neq b$ we write $a > b$. Beware that if the relation is not total, it is not true in general that $\neg(a \leq b)$ is equivalent to $a > b$.* [1Y7]

See in this regard the exercise 3.d.3.

Definition 3.c.7. *A total order* †26 *on a set X is said to be a* well ordering *if every non-empty subset of X has minimum.* [1X0]

Exercises

E3.c.8 Prerequisites:3.c.2. For each set A and each relation R between elements of A, explain if it is reflective, symmetric, antisymmetric and/or transitive; if it is a order relation, determine if it is total. [1WH] *(Proposed on 2022-12)*

- In $A = \mathbb{N} \setminus \{0\}$, nRm iff the greatest common divisor between n and m is 1
- In $A = \mathbb{N} \setminus \{0\}$, nRm if and only if n divides m
- In $A = \mathbb{N} \setminus \{0\}$, nRm if and only if $2n$ divides m
- In $A = \mathcal{P}(\mathbb{N})$, aRb if and only if $a \subseteq b$.

E3.c.9 Let $f : A \to B$ be a function, let $\sim$ be an equivalence relation on B: prove that the relation R between elements of A given by [1WK]

$$xRy \iff f(x) \sim f(y)$$

is an equivalence relation.

E3.c.10 Prerequisites:3.c.2,3.c.4. [224]

Given two relations $a \leq b$ and $a < b$ for $a, b \in A$, show that these are equivalent:

- $a \leq b$ is a (possibly partial) order relation and we identify
$$a < b = (a \leq b \wedge a \neq b) \quad ;$$
- $a < b$ is an irreflexive and transitive relation and $\forall x, y \in A$ at most one of $x < y$, $x = y$, $y < x$ holds; and we identify
$$a \leq b = (a < b \vee a = b) \quad .$$

This latter $a < b$ is called **strict (partial) order**.

E3.c.11 Prerequisites:3.c.2,3.c.4,3.c.10. Given two relations $a \leq b$ and $a < b$ for $a, b \in A$ show that these are equivalent: [24K]

- $a \leq b$ is a total order relation and
$$a < b = (a \leq b \wedge a \neq b) \quad ,$$
- $a < b$ is an irreflexive, trichotomous and transitive relations and
$$a \leq b = (a < b \vee a = b) \quad .$$

This latter $a < b$ is called **strict total order**.

E3.c.12 Consider $A = \mathbb{R}^2$ and consider the relations [1YH]

$$(x, y) \sim (x', y') \iff (x - x' \in \mathbb{Z} \wedge y - y' \in \mathbb{Z})$$

between elements of $\mathbb{R}^2$:

- show that it is an equivalence relation;
- graphically represent equivalence classes;
- describe the set $A/\sim$.

§3.d Order relations [1YY]

Let $(X, \leq)$ an orderered set, non-empty (*cf* definition 3.c.4)

Definition 3.d.1. *Given $x, y \in X$ remember that $x < y$ means $x \leq y \wedge x \neq y$.* [229]

- *When we have that $x \leq y$ or $y \leq x$ we will say that the two elements are* "comparable". *Conversely if neither $x \leq y$ nor $y \leq x$ then we will say that the two elements are* "incomparable".
- *An element $m \in X$ is called* maximal *if there is no element $z \in X$ such that $m < z$.*
- *An element $m \in X$ is called* minimal *if there is no element $z \in X$ such that $z < m$.*
- *An element $m \in X$ is called* maximum, *or* greatest element, *if, for any element $z \in X$, $z \leq m$.*
- *An element $m \in X$ is called* minimum, *or* least element, *if, for any element $z \in X$, $z \leq m$.*

Note that the definitions of minimum/minimal *can be obtained from* maximum/maximal *by reversing the order relation (and vice versa).*

Exercises

E3.d.2 Given an ordered set, show that the maximum, if it exists, is unique. [1WJ]

E3.d.3 Show that for any two $x, y \in X$ one of the following (mutually exclusive) cases holds [067] *(Solved on 2022-10-13)*

- $x = y$,
- $x < y$,
- $x > y$,
- x, y are incomparable.

Hidden solution: *[UNACCESSIBLE UUID '068']*

E3.d.4 Show that if $x < y \wedge y \leq z$ or $x \leq y \wedge y < z$ then $x < z$. [29D]

E3.d.5 Show that $m \in X$ is *maximal* if and only if *"for every $z \in X$ you have that $z \preceq m$ or z, m are incomparable"*. [069]

Hidden solution: [UNACCESSIBLE UUID '06B']

E3.d.6 Let $f : A \to B$ be a function, let $\preceq$ an order relation on B; consider the relation R between elements of A given by [1WM]

$$xRy \iff f(x) \preceq f(y) \quad ;$$

is it an order relation? What if we also assume that f is injective?

E3.d.7 Show that, if every non-empty subset admits minimum, then the order is total. [1WN]

E3.d.8 Consider $A = \mathbb{R}^2$ and consider the relation [1YJ]

$$(x, y) \preceq (x', y') \iff (x \le x' \land y \le y')$$

- show that it is an order relation; is it partial or total?
- Define $B = \{(x, y) : x^2 + y^2 \le 1\}$, let's consider it as an ordered set with the sorting $\preceq$: are there maxima? minima? maximals? minimals?

E3.d.9 Let $(X, \le)$ be a finite and ordered non-empty set then it has maximals and minimals. *Hidden solution:* [UNACCESSIBLE UUID '06D'] [06C] *(Proposed on 2022-12)*

E3.d.10 Build an order $\preceq$ on $\mathbb{N}$ with this property: for each $n \in \mathbb{N}$ [06F]

- the set $\{k \in \mathbb{N}, k \ne n, k \preceq n\}$ of the elements preceding n has exactly two maximals,
- the set $\{k \in \mathbb{N}, k \ne n, n \preceq k\}$ of the elements following n has exactly two minimals.

E3.d.11 Let X be a non-empty set and $R \subseteq X^2$ an order relation, then there is a total order T that extends R (i.e. $R \subseteq T$, considering relations as subsets of X^2). [06J] *(Proposed on 2022-12)*

E3.d.12 Prerequisites:4.b.1,3.b.43. Let X be ordered (partially). Show that these are equivalent [263] *(Solved on 2022-10-13)*

1. in each non-empty subset $A \subseteq X$ there is at least one minimal element;
2. there are no strictly decreasing functions $f : \mathbb{N} \to X$.

Hidden solution: [UNACCESSIBLE UUID '07Y']

See also Proposition 3.g.3.

§3.d.a Direct and filtering order [2FJ]

Definition 3.d.13. *Let $(X, \le)$ be a (partially) ordered set, we will say that it is **filtering*** †27 *if* [06M] *(Solved on 2022-11-24)*

$$\forall x, y \in X\ \exists z \in X,\ x < z \land y < z \quad . \tag{3.d.14}$$

The sets $\mathbb{R}, \mathbb{N}, \mathbb{Q}, \mathbb{Z}$ endowed with their usual order relations, are filtering.

Definition 3.d.15. *A **directed set** is an ordered set $(X, \le)$ for which* [06N]

$$\forall x, y \in X\ \exists z \in X,\ x \le z \wedge y \le z \quad . \tag{3.d.16}$$

Obviously a filtering order is direct.

Remark 3.d.17. *We have added the antisymmetric property to the usual definition of "Directed Set", see [14] (or other references in [38]).* [0NB]

This choice simplify the discussion (in particular it eases the use of concepts already used in the theory of ordered sets, such as maximum *and* maximal*); at the same time, by 7.d.3, this choice does not hinder the usufulness and power of the theory developed in this Section and in Section §7.d.*

Definition 3.d.18. *Given a directed set $(X, \le_X)$ a subset of it $Y \subseteq X$ is called **cofinal** if* [06P]

$$\forall x \in X\ \exists y \in Y,\ y \ge_X x \tag{3.d.19}$$

*More in general, another directed set $(Z, \le_Z)$ is said to be **cofinal** in X if there exists a map $i : Z \to X$ monotonic weakly increasing and such that $i(Z)$ is is cofinal in X;* i.e.

$$(\forall z_1, z_2 \in Z, z_1 \le_Z z_2 \Rightarrow i(z_1) \le_X i(z_2)) \ \wedge\ (\forall x \in X\ \exists z \in Z,\ i(z) \ge_X x) \tag{3.d.20}$$

(This second case generalizes the first one, where we may choose $i : Y \to X$ to be the injection map, and $\le_Y$ to be the restriction of $\le_X$ to Y.)

Definition 3.d.21. *If X is filtering,* "a neighborhood of ∞ in X" *is a subset $U \subseteq X$ such that* [231]

$$\exists k \in X \forall j \in X, j \ge k \Rightarrow j \in U\ .$$

Exercises

E3.d.22 Let $(X, \le)$ be a filtering ordered set, prove that it is an infinite set. *Hidden solution:* *[UNACCESSIBLE UUID '06R']* [06Q] *(Proposed on 2022-11)*

E3.d.23 Let $(X, \le)$ be a directed set: show that if there is a maximal element in X then it is the maximum. *Hidden solution:* *[UNACCESSIBLE UUID '06T']* [06S] *(Solved on 2022-10-27)*

E3.d.24 `Prerequisites:3.d.13,3.d.23`. Let $(X, \le)$ be a directed set. Show that these properties are equivalent: [06V] *(Solved on 2022-11-24)*

- $(X, \le)$ satisfies the *filtering property* (3.d.14),
- $(X, \le)$ has no maximum,
- $(X, \le)$ has no maximals.

Hidden solution: *[UNACCESSIBLE UUID '06W']*

E3.d.25 `Prerequisites:3.d.18`.Let $(X, \le)$ be a directed set, and $Y \subseteq X$ cofinal: show that $(Y, \le_{|_Y})$ is a directed set. [06X]

Similarly, if $(X, \le)$ is filtering, show that $(Y, \le_{|_Y})$ it is filtering.

E3.d.26 Prerequisites:3.d.21.Given $U_1, U_2 \subseteq J$ two neighborhoods of ∞ show that the intersection $U_1 \cap U_2$ is a neighborhood of ∞. *Hidden solution:* *[UNACCESSIBLE UUID '236']* . [232]

E3.d.27 Prerequisites:3.d.18,3.d.21.If $(X, \leq)$ a filtering set, $Y \subseteq X$ is cofinal, and $U \subseteq X$ is a neighborhood of ∞ in X, show that $U \cap Y$ is a neighborhood of ∞ in Y. *Hidden solution:* *[UNACCESSIBLE UUID '235']* [234]

A directed ordered set $(X, \leq)$ is a framework in which we can generalize the notion seen in 4.g.1.

Definition 3.d.28. *Let* P(x) *be a logical proposition that depends on a free variable* $x \in X$. *We will say that* [06Y] *(Solved on 2022-10-27)*

P(x) *holds eventually for* $x \in X$ *if*	$\exists y \in X, \forall x \in X, x \geq y \Rightarrow$ P(x) *holds;*
P(x) *frequently applies for* $x \in X$ *if*	$\forall y \in X, \exists x \in X, x \geq y$ *such that* P(x) *holds.*

Exercises

E3.d.29 The 4.g.5 property reformulates in this way. [070] *(Proposed on 2022-10-27)*

Show that «*P(x)* frequently applies for $x \in X$» if and only if the set

$$Y = \{x \in X : P(x)\}$$

is cofinal in X.

E3.d.30 Prerequisites:3.d.28,3.d.26.Show that «*P(x)* eventually holds for $x \in X$» if and only if the set [233]

$$U = \{x \in X : P(x)\}$$

is a neighborhood of ∞ in X.

E3.d.31 Prove that the properties 4.g.3, 4.g.4, 4.g.6 and 4.g.7 seen in Sec. §4.g also apply in this more general case 3.d.28. [06Z]

E3.d.32 Suppose that on the set X there is a relation R that is reflexive and transitive and satisfies [2B2]

$$\forall x, y \in X \ \exists z \in X, \ xRz, yRz \quad . \tag{3.d.33}$$

(as seen in (3.d.16))

This pair (X, R) is a "Directed Set" according to the usual definition (see [14] or other references in [38]).

Show that there exists another relation $\leq$ such that

- $\leq$ is a partial order and it satisfies (3.d.16);
- R extends $\leq$ that is;
$$\forall x, y \in X \ x \leq y \Rightarrow \ xRy \quad ;$$
- moreover $(X, \leq)$ is cofinal in (X, R).

Hidden solution: *[UNACCESSIBLE UUID '2GM']*

Further exercises on the subject are 6.a.2,8.15, and in Section §7.d.

†26 Actually the condition of *well ordering* for an order implies that the order is total; we leave it as an exercise 3.d.7.

†27 As defined in Definition 4.2.1 of the notes [2]. It is also called *strongly directed set*.

§3.d.b Lexicographic order [2FH]

Definition 3.d.34. *Given two ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, setting $Z = X \times Y$, we define the* ***lexicographic order*** *$\leq_Z$ on Z; let $z_1 = (x_1, y_1) \in Z$ and $z_2 = (x_2, y_2) \in Z$, then:* [071]

- *in the case $x_1 \neq x_2$, then $z_1 \leq_Z z_2$ if and only if $x_1 \leq_X x_2$;*
- *in the case $x_1 = x_2$, then $z_1 \leq_Z z_2$ if and only if $y_1 \leq_Y y_2$.*

This definition is then extended to products of more than two sets: given two vectors, if the first elements are different then we compare them, if they are equal we compare the second elements, if they are equal the thirds, etc.

Exercises

E3.d.35 Verify that $\leq_Z$ is an order relation. [1WP]

E3.d.36 If $(X, \leq_X)$ and $(Y, \leq_Y)$ are total orders, show that $(Z, \leq_Z)$ is a total order. [072]

E3.d.37 If $(X, \leq_X)$ and $(Y, \leq_Y)$ are well ordered, show that $(Z, \leq_Z)$ is a well ordering. [073]

E3.d.38 Let $X = \mathbb{N}^{\mathbb{N}}$ be ordered with lexicographic order. Build a function $f : X \to \mathbb{R}$ that is strictly increasing. *Hidden solution:* *[UNACCESSIBLE UUID '075']* [074]

E3.d.39 Consider $X = \mathbb{R} \times \{0, 1\}$ ordered with lexicographic order. Show that there is no function $f : X \to \mathbb{R}$ strictly growing. *Hidden solution:* *[UNACCESSIBLE UUID '077']* [076]

§3.d.c Total order, sup and inf [2FM]

Let $\leq$ a total order on a non-empty set X.

Definition 3.d.40. *Let $A \subseteq X$. The* ***majorants*** *of A (or* ***upper bounds****) are* [22R]

$$M_A \stackrel{def}{=} \{x \in X : \forall a \in A, a \leq x\} \quad .$$

A set A is ***bounded above*** *when there exists an $x \in X$ such that $\forall a \in A, a \leq x$, i.e. exactly when $M_A \neq \emptyset$.*

If M_A has minimum s, then s is th ***supremum****,* a.k.a. ***least upper bound****, of A, and we write $s = \sup A$.*

By reversing the order relation in the above definition, we obtain the definition of ***minorants/lower bounds****,* ***bounded below****,* ***infimum/greatest lower bound****.*

Lemma 3.d.41. *Let $A \subseteq X$ be a not empty set. We recall these properties of the supremum.* [22S]

1. *If A has maximum m then $m = \sup A$.*
2. *Let $s \in X$. We have $s = \sup A$ if and only if*
   - *for every $x \in A$ we have $x \leq s$.*
   - *for every $x \in X$ with $x < s$ there exists $y \in A$ with $x < y$.*

This last property is of very wide use in the analysis!

The proof is left as a (useful) exercise. Hidden solution: [UNACCESSIBLE UUID '22T']

Exercises

E3.d.42 Let B be a non-empty set that is bounded from below, let L the set of minorants of B; we note that L is upper bounded, and suppose that $\alpha = \sup L$ exists: then $\alpha \in L$ and $\alpha = \inf B$. *Hidden solution:* *[UNACCESSIBLE UUID '079']* [078] *(Proposed on 2022-10-13)*

E3.d.43 Prerequisites:3.d.42. Show that if for the total ordering of X all "suprema" exist then all "infima" also exist; and vice versa. Precisely, show that these are equivalent: [07B] *(Proposed on 2022-10-13)*

- Every non-empty set bounded from below in X admits greatest lower bound;
- every non-empty set bounded from above in X admits least upper bound.

Hidden solution: *[UNACCESSIBLE UUID '207']*

§3.d.d Total ordering, intervals [2DW]

Let $\le$ a total order on a non-empty set X.

Definition 3.d.44. *A set $I \subseteq X$ is an* ***interval*** *if for every $x, z \in I$ and every $y \in X$ with $x < y < z$ we have $y \in I$.* [07C]

Note that the empty set is an interval.

Definition 3.d.45. *Given $x, z \in X$ the following standard intervals are defined* [07D]

$$
\begin{aligned}
(x,z) &= \{y \in X : x < y < z\} \\
(x,z] &= \{y \in X : x < y \le z\} \\
(x,\infty) &= \{y \in X : x < y\} \\
[x,z) &= \{y \in X : x \le y < z\} \\
[x,z] &= \{y \in X : x \le y \le z\} \\
[x,\infty) &= \{y \in X : x \le y\} \\
(-\infty,z) &= \{y \in X : y < z\} \\
(-\infty,z] &= \{y \in X : y \le z\} \\
(-\infty,\infty) &= X \ .
\end{aligned}
$$

Note that there are 9 cases, 3 for the LHS and 3 for the RHS. We concord that $\infty, -\infty$ are symbols and not elements of X; if X has a maximum m then the intervals are preferably written as $(x,\infty) = (x,m]$ and $[x,\infty) = [x,m]$; similarly if X has a minimum. [24Y]

Exercises

E3.d.46 Prerequisites:3.d.44,3.d.45,3.b.23. [07F]

Let $\mathcal{F}$ be a non-empty family of intervals.

Show that the intersection $\bigcap \mathcal{F}$ of all intervals is an interval.

Suppose the intersection $\bigcap \mathcal{F}$ is not empty, show that the union $\bigcup \mathcal{F}$ is an interval.

Hidden solution: *[UNACCESSIBLE UUID '07G']*

E3.d.47 Prerequisites:3.d.44,3.d.45. [07H] (Solved on 2022-10-13)

Find an example of a set X with total ordering, in which there is an interval I that does not fall into any of the categories viewed in 3.d.45.

Hidden solution: *[UNACCESSIBLE UUID '07J']*

E3.d.48 Prerequisites:3.d.44,3.d.45,3.d.46. [07K] (Proposed on 2022-10-13)

Let $A \subseteq X$ be a non-empty set; let$I$ the smallest interval that contains A; this is defined as the intersection of all intervals that contain A (and the intersection is an interval, by 3.d.46). Let M_A be the family of majorants of A, M_I of I; show that $M_A = M_I$. In particular A is bounded from above if and only I is bounded from above; if moreover A has supremum, then $\sup A = \sup I$. (Similarly for the minorants and infimum). *Hidden solution:* *[UNACCESSIBLE UUID '07M']*

E3.d.49 Prerequisites:3.d.44,3.d.45,3.d.46,3.d.48. [07N]

Let X be a totally ordered set. Show that the following two are equivalent.

- Every $A \subseteq X$ non-empty bounded from above and from below admits supremum and infimum.
- Each non-empty interval $I \subseteq X$ falls in one of the categories seen in 3.d.45.

Hidden solution: *[UNACCESSIBLE UUID '07P']*

E3.d.50 Prerequisites:3.d.44,3.d.45,3.d.46.Difficulty:*. [206]

At the beginning of the section we assumed that the ordering $\leq$ on X be total. The definitions of *interval* in 3.d.44 and 3.d.45 however, they can also be given for an order that is not (necessarily) total. What happens in exercise 3.d.46 when the order is not total? Which result is true, which is false, and if so what counterexample can we give?

§3.d.e Order types

Definition 3.d.51. *Given two ordered non-empty sets $(X, \leq_X)$ and $(Y, \leq_Y)$, we will say that* "they have the same order type"*, or* "order-isomorphic"*, or more briefly that they are* "equiordinate"[†28]*, if there is a strictly increasing monotonic bijective function $f : X \to Y$, whose inverse f^{-1} is strictly increasing. The function f is the* "order isomorphism". [07V]

Remark 3.d.52. *Note that if $(X, \leq_X)$ and $(Y, \leq_Y)$ are* equiordinate *then X and Y are* equipotents*; but given an infinite set X, there exist on it orders of different types — even if we consider only the well orders. (See for example exercise 3.i.20)* [21R] (Solved on 2021-11-18)

Remark 3.d.53. *Note that if two sets are equiordinate, then they enjoy the same properties: if one is totally ordered, so is the other; if one is well ordered, so is the other; etc etc .... See 3.d.55.* [21V]

Exercises

E3.d.54 Show that the relation *"having the same order type"* is an equivalence relation. Given a set X, let's consider all possible orders on X, the relation therefore defines equivalence classes, and each class is (precisely) an"*order type*" on X. [220] (Proposed on 2021-11-18)

[†28] The wording "equiordinate" is not standard.

E3.d.55 Given two ordered non-empty sets $(X, \leq_X)$ and $(Y, \leq_Y)$, and $f : X \to Y$ as defined in 3.d.51. [22P] (Proposed on 2023-01-17)

- If $A \subseteq X$ and $m = \max A$ then $f(m) = \max f(A)$; similarly for the minimums;
- $(X, \leq_X)$ is totally ordered if and only if $(Y, \leq_Y)$ is;
- $(X, \leq_X)$ is well ordered if and only if $(Y, \leq_Y)$ is.
- Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are well ordered, let S_X and respectively S_Y be the functions "successor", 3.i.7, then we have that x is not the maximum of X if and only if $f(x)$ is not the maximum of Y, and in this case $y = S_X(x)$ if and only if $f(y) = S_Y(f(x))$.

E3.d.56 Given two totally ordered non-empty sets $(X, \leq_X)$ and $(Y, \leq_Y)$, suppose there exists a strictly increasing monotonic bijective function $f : X \to Y$: show that then its inverse f^{-1} is strictly increasing, and consequently $(X, \leq_X)$ and $(Y, \leq_Y)$ are equiordinate. *Hidden solution:* *[UNACCESSIBLE UUID '21T']* [21P]

E3.d.57 Find a simple example of two non-empty (partially) ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, for which there exists a strictly increasing monotonic bijective function $f : X \to Y$, whose inverse f^{-1} is not strictly increasing. *Hidden solution:* *[UNACCESSIBLE UUID '21S']* [21Q]

§3.d.f Concatenation

Definition 3.d.58. *Given two ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, with X, Y disjoint,* ***the concatenation of*** *X* ***with*** *Y is obtained defining $Z = X \cup Y$ and providing it with the ordering $\leq_Z$ given by:* [21W]

- *if $z_1, z_2 \in X$ then $z_1 \leq_Z z_2$ if and only if $z_1 \leq_X z_2$;*
- *if $z_1, z_2 \in Y$ then $z_1 \leq_Z z_2$ if and only if $z_1 \leq_Y z_2$;*
- *If $z_1 \in X$ and $z_2 \in Y$ then you always have $z_1 \leq_Z z_2$.*

This operation is sometimes denoted by the notation $Z = X + Y$.

If the sets are not disjoint, we can replace them with disjoint sets defined by $\tilde{X} = \{0\} \times X$ and $\tilde{Y} = \{1\} \times Y$, then we may "copy" the respective orders, and finally we can perform the concatenation of $\tilde{X}$ and $\tilde{Y}$.

Exercises

E3.d.59 Let $k \in \mathbb{N}$ and let $I = \{0, \dots, k\}$ with the usual ordering of $\mathbb{N}$: show that the concatenation of I with $\mathbb{N}$ has the same type of order as $\mathbb{N}$; while the concatenation of $\mathbb{N}$ with I does not have the same type of order. [21X]

E3.d.60 Prerequisites:3.d.34,3.d.51.Let $(X_1, \leq_1)$, $(X_2, \leq_2)$ be two disjoint and partially ordered sets and with the same order type. Let $I = \{1, 2\}$ with the usual order; let $Z = I \times X_1$ equipped with lexicographical order; Let W be the concatenation of of X_1 with X_2: show that Z and W have the same type of order. *Hidden solution:* *[UNACCESSIBLE UUID '221']* [21Y]

E3.d.61 Let X_1, X_2 be two disjoint and well-ordered sets. Let W be the concatenation of X_1 with X_2: show that it is well ordered. [21Z]

§3.e Functions [1YR]

The definition of *function* can be obtained from set theory in this way.

Definition 3.e.1. *Given two sets A, B, a function $f : A \to B$ is a triple* [1Y6]

$$A, B, F$$

(where A is said domain *and B* codomain*) and F is a relation $F \subseteq A \times B$ such that*

$$\forall x \in A \exists! y \in B, xFy \quad ;$$

i.e. it enjoys the properties of being functional *and* total *(defined in 3.c.2).*

Being the element y unique, we can write $y = f(x)$ to say that y is the only element in relation xFy with x.

The set F is also called graph *of the function.*

Definition 3.e.2. *Given nonempty sets I, A, a* ***sequence with indexes in I and taking values in*** *A is a function $a : I \to A$; this though is usually written by the notation $(a_n)_{n\in I}$. To denote the codomain, the notation $(a_n)_{n\in I} \subseteq A$ is also employed. In this text, in most cases, we will have that $I = \mathbb{N}$, and in this case we will simply write (a_n).* [16G]

In practice, the definition of function is always written as $f : A \to B$; for this reason the graph is defined as

$$F = \{(a, b) \in A \times B : b = f(a)\} \quad .$$

Remark 3.e.3. *Let A be a non-empty set, let $f : A \to \{0, 1\}$ and $g : A \to \{1\}$ both given by $f(x) = g(x) = 1$ for each $x \in A$.* [08X]

Let F, G respectively be the graphs: note that $F = G$ (!) Will we say that $f = g$ or not? We choose "not", otherwise the concept of "surjective" would not make sense.

For this reason in the definition we decided that the function is the triple "domain", "codomain", "relation".

Exercises

E3.e.4 Show that the composition of two injective functions is an injective function. [1WQ]

E3.e.5 Show that the composition of two surjective functions is a surjective function. [1WR]

E3.e.6 Let $f : \mathbb{N} \to \mathbb{N}$ be an assigned function and I its image, prove that $A \subseteq \mathbb{N}$ exists such that $f|_A$ is injective and $f(A) = I$. *(Hint it may be useful to know that the usual order of $\mathbb{N}$ is a well-order* cf *3.i.1 and 4.d.4).* [1WS]

Hidden solution: *[UNACCESSIBLE UUID '1WT']*

Note: The result is true for any function $f : A \to B$, but the proof requires the axiom of choice.

E3.e.7 Let $I, J \subseteq \mathbb{R}$ and let $f : I \to J$ be given by $f(x) = \sin(x)$. By choosing $I = \mathbb{R}$ or $I = [0, \pi/2]$ or $I = [-\pi/2, \pi/2]$, and choosing $J = \mathbb{R}$ or $J = [-1, 1]$, say for which choices f is surjective, and for which it is injective. [08Y] *(Proposed on 2022-12)*

(This exercise is to make you ponder about the difference between "formula" and "function.".)

E3.e.8 Let $A, B \subseteq \mathbb{R}$ and let $f : A \to B$ be defined by the formula $f(x) = x^2$; tell if, for the following choices of A, B, the function f is injective and/or surjective. [1X3]

1. $A = \mathbb{R}, B = \mathbb{R}$
2. $A = \mathbb{R}, B = [0, \infty)$
3. $A = [0, \infty), B = \mathbb{R}$
4. $A = [0, \infty), B = [0, \infty)$

If the function is bijective, what is its inverse commonly called?

(This exercise is to make you ponder about the difference between ”formula” and ”function.”

E3.e.9 Given $f, g : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2 - 1$ and $g(n) = (n+1)^2$, write explicitly $f \circ g$ and $g \circ f$, say if they coincide or are different functions. [1X4]

E3.e.10 Find an example of $f, g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g \equiv g \circ f$, but neither f nor g are bijective. [1X5]

E3.e.11 Let $f : \mathbb{R} \to \mathbb{R}$ be bijective, and $F \subseteq \mathbb{R}^2$ its graph; let f^{-1} be the inverse of f and let G be its graph; show that G is the symmetric of F with respect to the bisector of the first and third quadrants. [1X6]

E3.e.12 Let D, C be non-empty sets and $f : D \to C$ a function. Let I a non-empty family of indexes, $B_i \subseteq C$ for $i \in I$. Given $B \subseteq C$ remember that the **counterimage** of B is [091]

$$f^{-1}(B) \stackrel{\text{def}}{=} \{x \in D, f(x) \in B\} \ ,$$

Given $B \subseteq C$ we write $B^c = \{x \in C, x \notin B\}$ to denote the complement. Show these counterimage properties.

$$f^{-1}\big(\bigcup_{i \in I} B_i\big) = \bigcup_{i \in I} f^{-1}(B_i) \tag{3.e.13}$$

$$f^{-1}\big(\bigcap_{i \in I} B_i\big) = \bigcap_{i \in I} f^{-1}(B_i) \tag{3.e.14}$$

$$f^{-1}(B^c) = f^{-1}(B)^c \ . \tag{3.e.15}$$

E3.e.16 Let D, C be non-empty sets and $f : D \to C$ a function. Let I be a non-empty family of indexes, $A_i \subseteq D$, for $i \in I$. Given $A \subseteq D$ remember that the **image** of A is the subset $f(A)$ of D given by [092]

$$f(A) \stackrel{\text{def}}{=} \{f(x), x \in A\} \ .$$

Show these image properties.

$$f\big(\bigcup_{i \in I} A_i\big) = \bigcup_{i \in I} f(A_i)$$

$$f\big(\bigcap_{i \in I} A_i\big) \subseteq \bigcap_{i \in I} f(A_i) \ .$$

Show that the function is injective if and only if

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \tag{3.e.17}$$

is an equality for every choice of $A_1, A_2 \subseteq D$.

E3.e.18 Let D, C be non-empty sets and $f : D \to C$ a function. Given $U \subseteq C$ show that [250]

$$f(f^{-1}(U)) \subseteq U\,;$$

if f is surjective show that they are equal; find an example where they are different.

E3.e.19 Let D, C be non-empty sets and $f : D \to C$ a function. Given $A \subseteq D$ show that [251]

$$f^{-1}(f(A)) \supseteq A\,;$$

if f is injective show that they are equal; find an example where they are different.

E3.e.20 Let A, B be non-empty sets. [2BX]

- Suppose that $f : A \to B$ is an injective function: there exists a surjective function $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$ (the identity function). (Such g is a *left inverse* of g).
- Suppose that $g : B \to A$ is a surjective function: there exists a injective function $f : A \to B$ such that $g \circ f = \mathrm{Id}_A$. (Such f is a *right inverse* of g).

The proof of the second statement requires the Axiom of Choice (see 3.b.46).

Vice versa.

- If $f : A \to B$ has a *left inverse* then it is an injective function.
- If $g : B \to A$ has a *right inverse*, then it is a surjective function.

Hidden solution: *[UNACCESSIBLE UUID '2BY']*

E3.e.21 Let A be a set and let $g : A \to A$ be injective. We define the relation $x \sim y$ which is true when an $n \geq 0$ exists such that $x = g^n(y)$ or $x = g^n(y)$; where [093]

$$g^n = \overbrace{g \circ \cdots \circ g}^{n}$$

is the n-th iterate of the composition. (We decide that g^0 is identity). Show that $x \sim y$ is an equivalence relation. Study equivalence classes. Let $U = \bigcap_{n=1}^{\infty} g^n(A)$ be the intersection of repeated images. Show that each class is entirely contained in U or is external to it.

Hidden solution: *[UNACCESSIBLE UUID '094']*

E3.e.22 Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x$. Is there a continuous function for which $f(f(x)) = -x$? (Hint: show that for every such f you have $f^{-1}(\{0\}) = \{0\}$). *Hidden solution:* *[UNACCESSIBLE UUID '096']* [095]

E3.e.23 Show that there exists a function $f : [0,1] \to [0,1]$ such that $f(f(x)) = \sin(x)$. Is there a continuous function? *Hidden solution:* *[UNACCESSIBLE UUID '098']* [097]

E3.e.24 Let D, C be non-empty sets. A **partial function** from D in C is a function $\varphi : B \to C$ where $B \subseteq D$. (The definition of "*function*" is in 3.e.1). [01P] *(Solved on 2022-11-15)*

It can be convenient to think of the partial function as a relation $\Phi \subseteq D \times C$ such that, if $(x, a), (x, b) \in \Phi$ then $a = b$ (see 3.c.2). The two notions are equivalent in this sense: given Φ we build the domain of φ, which we will call B, with the projection of Φ on the first factor i.e. $B = \{x \in D : \exists c \in C, (x, c) \in \Phi\}$, and we define $\varphi(x) = c$ as the only element $c \in C$ such that $(x, c) \in \Phi$; vice versa Φ is the graph of φ.

Partial functions, seen as relations Φ, are of course sorted by inclusion; equivalently $\varphi \le \psi$ if $\varphi : B \to C$ and $\psi : E \to C$ and $B \subseteq E \subseteq D$ and $\varphi = \psi_{|B}$.

Let now U be a **chain**, i.e. family of partial functions that is totally ordered according to the order previously given; seeing each partial function as a relation, let Ψ be the union of all relations in U; show that Ψ is the graph of a partial function $\psi : E \to C$, whose domain E is the union of all the domains of the functions in U, and whose image I is the union of all images of functions in U

If moreover all functions in U are injective, show that ψ is injective.

Hidden solution: [UNACCESSIBLE UUID '01Q']

§3.f Elementary functions

Exercises

E3.f.1 Let n, m, k be positive integers. Prove that the number $(n+\sqrt{m})^k + (n-\sqrt{m})^k$ is integer. [09G]

Hidden solution: [UNACCESSIBLE UUID '09H']

E3.f.2 Let K be a positive integer, N an integer, and $I = \{N, N+1, \dots, N+K\}$ be the sequence of integers from N to $N+K$. For each $n \in I$ we set an integer values a_n. Let p be the only one polynomial of degree K such that $p(n) = a_n$ for every $n \in I$. [09J]

- Show that p has rational coefficients.
- Show that $p(x)$ is integer for every x integer.
- Find an example of a polynomial p which takes integer values for x integer, but not all coefficients of p are integers.
- What happens if I contains $K+1$ integers, but not consecutive? Is it still true that, defining $p(x)$ as above, p only assumes integer values on integers?

E3.f.3 Let $p(x)$ be a polynomial with real coefficients of degree n, show that exists $c > 0$ such that for every x we have $|p(x)| \le c(1+|x|^n)$. *Hidden solution:* [UNACCESSIBLE UUID '09M'] [09K]

E3.f.4 Prove that, for $n \ge 2$, [211] *(Proposed on 2022-12)*

$$\sum_{k=1}^{n-1} \frac{1}{k} \ge \log(n)$$

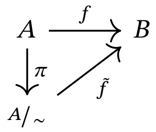
Hidden solution: [UNACCESSIBLE UUID '212']

§3.g Projecting to the quotient [1Z5]

Definition 3.g.1. *Let A be a set and $\sim$ an equivalence relation. We denote by* [23M]

$$A/_\sim$$

*the quotient space, that is, the set of all equivalence classes; the **canonical projection** is the map $\pi : A \to A/_\sim$ that associates each $x \in A$ with the class $[x] \in A/\sim$.*



[UNACCESSIBLE UUID '20Q']

Proposition 3.g.2. [1Z6]

- *Suppose that the function $f : A\times A \to B$ is invariant for the equivalence relation $\sim$ in all its variables, i.e.*

$$\forall x, y, v, w \in A, \quad x \sim y \wedge v \sim w \Rightarrow f(x, v) = f(y, w) \quad ;$$

let $\tilde f$ be the projection to the quotient $\widetilde{f} : {}^{A}/_{\sim} \times {}^{A}/_{\sim} \to B$ that satisfies

$$f(x, y) = \widetilde{f}(\pi(x), \pi(y)) \quad .$$

If f is commutative (resp. *associative) then $\widetilde{f}$ is commutative* (resp. *associative).*

- *If R is a relation in $A \times A$ invariant for $\sim$, and R is reflexive* (resp *symmetrical, antisymmetric, transitive) then $\widetilde{R}$ is reflexive* (resp *symmetrical, antisymmetric, transitive).*

- *If A and B are ordered and the order is invariant, and f is monotonic, then $\widetilde{f}$ is monotonic.*

Proposition 3.g.3. *(Replaces 06G) (Replaces 06H) Consider R a transitive and reflexive relation in $A \times A$; such a relation is called a* preorder *[43]; we define $x \sim y \iff (xRy \wedge yRx)$ then $\sim$ is an equivalence relation, R is invariant for $\sim$, and $\widetilde{R}$ (defined as in 3.g.2) is an order relation.* [1Z7]

Proof. 1. $\sim$ is clearly reflexive and symmetrical; is transitive because if $x \sim y, y \sim z$ then $xRy \wedge yRx \wedge yRz \wedge zRy$ but being R transitive you get $xRz \wedge zRx$ i.e. $x \sim z$

2. Let $x, y, \tilde x, \tilde y \in X$ be such that $x \sim \tilde x, y \sim \tilde y$ then we have $xR\tilde x \wedge \tilde x Rx \wedge yR\tilde y \wedge \tilde y Ry$ if we add xRy, by transitivity we get $\tilde x R\tilde y$; and symmetrically.

3. Finally, we see that $\widetilde{R}$ is an order relation on Y. Using the (well posed) definition "$[x]\widetilde{R}[y] \iff xRy$" we deduce that $\widetilde{R}$ is reflexive and transitive (as indeed stated in the previous proposition). $\widetilde{R}$ is also antisymmetric because if for $z, w \in A/\sim$ you have $z\widetilde{R}w \wedge w\widetilde{R}z$ then taken $x \in z, y \in w$ we have $xRy \wedge yRx$ which means $x \sim y$ and therefore $z = w$.

□

Exercises

E3.g.4 $\mathbb{Z}$ are the relative integers with the usual operations. Let $p \geq 1$ a fixed integer. Consider the equivalence relation [1Z8]

$$n \sim m \iff p|(n - m)$$

that is, they are equivalent when $n - m$ is divisible by p.

Show that there are p equivalence classes $[0], [1], \dots [p-1]$ We indicate the quotient space with $\mathbb{Z}/(p\mathbb{Z})$ or more briefly $\mathbb{Z}_p$.

Show that the usual operations of sum, subtraction, product in $\mathbb{Z}$ pass to the quotient.

§3.h Natural numbers in ZF [246]

In this section we will build a model of the natural numbers inside the ZF set theory; this model satisfies Peano's axioms 4.3 and has an order relation that satisfies 4.d.1, so this model enjoys all properties described in Sec. §4; for this reason in this section we will mostly discuss properties that are specific of this model.

§3.h.a Successor

Definition 3.h.1. *Given x the **successor** is defined as* [24X]

$$S(x) \stackrel{def}{=} x \cup \{x\} \quad . \tag{3.h.2}$$

We will often write Sx instead $S(x)$ to ease notations.

*We say that a set A is **S-saturated** if $\emptyset \in A$ and if for every $x \in A$ you have $S(x) \in A$.* †29

Exercises

E3.h.3 Note that $z \in S(x)$ if and only if $z \in x \vee z = x$; [24V]

E3.h.4 Prerequisites:3.b.36,3.h.3.Prove that $x \in S(x)$ and $x \subsetneq S(x)$. *Hidden solution:* *[UNACCESSIBLE UUID '24N']* [24M]

E3.h.5 Prerequisites:3.b.36,(3.h.2),3.h.3.Let x, y be elements (generic, not necessarily natural numbers), such that [239]

$$x \subseteq y \subseteq S(x) \tag{3.h.6}$$

prove that

$$x = y \vee y = S(x) \quad ;$$

where the above two are mutually exclusive, and (in the hypothesis (3.h.6) above) the second one holds if and only if $x \in y$; summarizing

$$(3.h.6) \Rightarrow (x = y \iff y \neq S(x) \iff x \notin y) \quad .$$

Note the analogy with 3.i.9.

E3.h.7 Prove that the intersection of S-saturated sets provides an S-saturated set. [245]

E3.h.8 Prerequisites:3.b.36,(3.h.2),3.h.3. †30 Prove that [1YM]

$$x = y \iff S(x) = S(y) \quad .$$

In particular this shows that, if A is an S-saturated set, then the function $S : A \to A$ is well defined, and its graph is the relation

$$\{(x, y) \in A^2 : y = S(x)\} \quad ;$$

moreover S is injective.

Hidden solution: *[UNACCESSIBLE UUID '1YN']*

E3.h.9 Find an example of x, y such that $x \in y \land x \subseteq y$ but $S(x) \notin S(y)$ [24Q]

Hidden solution: [UNACCESSIBLE UUID '24R']

E3.h.10 Show that when $x \in y \land x \subseteq y$ then $S(x) \subseteq y$. *Hidden solution:* [UNACCESSIBLE UUID '24T'] [24S]

§3.h.b Natural numbers in ZF

Definition 3.h.11. *The **axiom of infinity** guarantees that there is a set A that is S-saturated.* [243]

Using the axiom of infinity 3.h.11 we can prove the existence of the set of natural numbers.

Theorem 3.h.12. *$\mathbb{N}$ is the smallest S-saturated set.* [244]

Proof. Given a set A that is S-saturated, $\mathbb{N}_A$ is defined as the intersection of all S-saturated subsets of A. By 3.h.7, $\mathbb{N}_A$ is S-saturated. Given two sets A, B that are S-saturated, it is proven that $\mathbb{N}_A = \mathbb{N}_B$: we denote then by $\mathbb{N}$ this set. In particular, given a set A that is S-saturated, we have $\mathbb{N} \subseteq A$. □

Example 3.h.13. *In this model, the first natural number* 0 *is identified with* $\emptyset$. *Then* [291]

$$\begin{aligned}
1 &= 0 \cup \{0\} = \{0\} = \big\{\{\}\big\},\\
2 &= 1 \cup \{1\} = \{0, 1\} = \Big\{\{\}, \big\{\{\}\big\}\Big\},\\
3 &= 2 \cup \{2\} = \{0, 1, 2\} = \bigg\{\{\}, \big\{\{\}\big\}, \Big\{\{\}, \big\{\{\}\big\}\Big\}\bigg\},\\
&\dots.
\end{aligned}$$

Remark 3.h.14. *This fact holds true:* [25C]

$$\forall y \in \mathbb{N}, y \neq \emptyset \Rightarrow \exists x \in \mathbb{N}, S(x) = y$$

this can be proven by induction, as in 4.2, or by proving that, if

$$\exists y \in \mathbb{N}, y \neq \emptyset \land \forall x \in \mathbb{N}, S(x) \neq y$$

then $\mathbb{N} \setminus \{y\}$ would be an S-saturated set smaller than $\mathbb{N}$, a contradiction. In particular by 3.h.8 we get that the successor function

$$S : \mathbb{N} \to \mathbb{N} \setminus \{0\}$$

is bijective.

*If $n \neq 0$, we will call $S^{-1}(n)$ the **predecessor** of n.*

We can also prove directly the induction principle.

Theorem 3.h.15 (Induction Principle)**.** *Let $A \supseteq \mathbb{N}$ and $P(n)$ be a logical proposition that can be evaluated for $n \in A$. Suppose the following two assumptions are satisfied:* [23B]

- *$P(n)$ is true for $n = 0$ and*

†29 In [13] such set is called *inductive*.
†30 Proposition 1.7.4 point 5 in [2].

- $\forall n \in \mathbb{N}, P(n) \Rightarrow P(S(n))$;

then P is true for every $n \in \mathbb{N}$.
The first hypothesis is known as "the basis of induction" and the second as "inductive step"

Proof. Let $U = \{n \in \mathbb{N} : P(n)\}$, we know that $0 \in U$ and that $\forall x, x \in U \Rightarrow S(x) \in U$, so U is S-saturated and $U \subseteq N$ we conclude that $U = \mathbb{N}$. □

Theorem 3.h.16. *Consider the order relation* $\subseteq$ *on* $\mathbb{N}$*; then* [24D]

$$\forall x, y \in \mathbb{N}, (x \subseteq Sy \wedge x \neq Sy) \iff (x \subseteq y) \quad . \tag{3.h.17}$$

This will be proven in Exercise 3.h.26.

To prove the above theorem, the exercises in the following section can be used.

Remark 3.h.18. Peano's Axioms *are provable in this model; moreover the order relation satisfies the requirements of Hypothesis 4.d.1; therefore this model of* $\mathbb{N}$ *enjoys all properties discussed in Sec. §4: all different versions of the induction principle;* $(\mathbb{N}, \subseteq)$ *is well-ordered; definitions by recursion; arithmetic, etc.* [26K]

When we will want to compare this model with other models, we will denote it by $\mathbb{N}_{ZF}$.

The ordered set $\mathbb{N}, \subseteq$ then enjoys these properties.

Proposition 3.h.19. *This model of* $\mathbb{N}$ *is a well-ordered set with the ordering* [26J]

$$n \leq m \iff n \subseteq m \quad .$$

Moreover in this model we have

$$\forall n, m \in \mathbb{N}, n \in m \iff (n \subseteq m \wedge n \neq m) \quad . \tag{3.h.20}$$

so, defining (as usual)

$$n < m \doteq (n \leq m \wedge n \neq m)$$

we can write

$$n \in m \iff n < m \quad .$$

This is proven in the following exercises, see in particular 3.h.28.

More details are in the course notes (Chap. 1 Sec. 7 in [2]); or [13],[12].

§3.h.c Transitive sets

Definition 3.h.21. *A set A is said to be* ***transitive*** *if these equivalent conditions hold:* [24Z]

- $$\forall x, x \in A \Rightarrow x \subseteq A \quad ,$$
  i.e. every element of A is also a subset of A;
- $$\forall x, y, x \in y \wedge y \in A \Rightarrow x \in A \quad .$$

Example 3.h.22. *Examples of transitive sets are:* [290]

$$\begin{aligned} \{\} &= 0 \\ \{\{\}\} &= 1, \\ \{\{\},\{\{\}\}\} &= 2, \\ \{\{\},\{\{\}\},\{\{\{\}\}\}\}&, \\ \{\{\},\{\{\}\},\{\{\{\}\}\},\{\{\{\{\}\}\}\}\}&, \\ &\dots \\ \{\{\},\{\{\}\},\{\{\},\{\{\}\}\}\} &= 3 \\ \{\{\},\{\{\}\},\{\{\{\}\}\},\{\{\},\{\{\}\}\}\}&, \\ &\dots. \end{aligned}$$

Comparing with 3.h.13 note that there are transitive sets that are not natural numbers.

Exercises

E3.h.23 If every element of A is a transitive set then the relation $x \in y$ is a transitive relation in A . (Note that this holds also if A is not a transitive set.) *Hidden solution:* *[UNACCESSIBLE UUID '25K']* [25J]

E3.h.24 Prove that for each $m \in \mathbb{N}$, m is a transitive set. [257]

Hidden solution: *[UNACCESSIBLE UUID '258']*

E3.h.25 Prerequisites:3.h.21,3.h.10,3.h.24. [26N]

$\forall n, k \in \mathbb{N}$ if $n \in k$ then $Sn \subseteq k$.

(Hint: you do not need induction, use that each $n \in \mathbb{N}$ is a transitive set).

E3.h.26 Prerequisites:3.h.25,3.b.36.To assert the Theorem 3.h.16 we have to prove [26P]

$$\forall x, y \in \mathbb{N}, (x \subseteq Sy \wedge x \neq Sy) \iff (x \subseteq y) \quad . \tag{3.h.17}$$

Prove that if X is a set where each element is transitive

$$\forall x, y \in X, (x \subseteq Sy \wedge x \neq Sy) \iff (x \subseteq y) \quad . \tag{3.h.27}$$

Hidden solution: *[UNACCESSIBLE UUID '26Q']*

The previous exercises prove Theorem 3.h.16, then by results of Sec. §4 we obtain that $(\mathbb{N}, \leq)$ is well ordered.

Here following are other interesting exercises.

Exercises

E3.h.28 We know from 3.h.18 that the relation $n \subseteq m$ is total in $\mathbb{N}$. Prove that [269]

$$\forall n, m \in \mathbb{N}, n \in m \iff (n \subseteq m \land n \neq m) \quad . \tag{3.h.20}$$

By 3.c.11 this implies

$$\forall n, m \in \mathbb{N}, n \subseteq m \iff (n \in m \lor n = m) \quad .$$

Hidden solution: [UNACCESSIBLE UUID '26B']

E3.h.29 Prove this assertion [265]

$$\forall k \in \mathbb{N}, k \neq \emptyset \Rightarrow \emptyset \in k \quad .$$

Hidden solution: [UNACCESSIBLE UUID '266']

E3.h.30 Prerequisites:3.h.16,3.h.19.Prove that [25D]

$$\forall x, y \in \mathbb{N}\ ,\ x \subseteq y \ \land\ x \neq y \Rightarrow Sx \subseteq y \quad .$$

Hidden solution: [UNACCESSIBLE UUID '279']

E3.h.31 Having fixed $N \in \mathbb{N}$, consider the ordering $n \subseteq m$ for $n, m \in N$. Since $N \subseteq \mathbb{N}$ is well ordered, then Proposition 3.h.19 implies that $(N, \subseteq)$ is well ordered; nonetheless prove directly by induction that $n \subseteq m$ is a well ordering in N. [25W]

Hidden solution: [UNACCESSIBLE UUID '25X']

E3.h.32 Prove that [25Z]

$$\bigcup \mathbb{N} = \mathbb{N} \quad .$$

Hidden solution: [UNACCESSIBLE UUID '260']

§3.h.d Ordinals

Perusing the above results we can give some elements of the theory of ordinals.

Definition 3.h.33. *An **ordinal** (according to Von Neumann) is a transitive set A such that any element in A is a transitive set.*

Exercises

E3.h.34 Prerequisites:3.b.36,3.c.10.If A is a set where $\in$ is transitive, we define [25Q]

$$x \leq y \doteq x \in y \lor x = y$$

prove that $x \leq y$ is a (possibly partial) order relation in A.

Hidden solution: [UNACCESSIBLE UUID '25S']

E3.h.35 Prove that the intersection of transitive sets is a transitive set. [25B]

E3.h.36 Prove that the intersection of ordinals is an ordinal. [25N]

E3.h.37 Prove that if X is an ordinal and $A \in X$ then A is an ordinal. [25M]

Hidden solution: [UNACCESSIBLE UUID '25P']

E3.h.38 Use the axiom of foundation 3.b.36 to prove that if A is transitive and $A \neq \emptyset$ then $\emptyset \in A$. [25G]

Hidden solution: [UNACCESSIBLE UUID '25H']

E3.h.39 Prove that $\mathbb{N}$ is a transitive set. (Hint: use induction.) [255]

Hidden solution: [UNACCESSIBLE UUID '256']

This and 3.h.24 say that $\mathbb{N}$ is an ordinal.

E3.h.40 Let X be an ordinal and [26S]

$$P_x = \{z \in X, z \in x\}$$

show that

$$\forall x, y \in X \ , \ P_x = P_y \Rightarrow x = y \quad .$$

Hidden solution: [UNACCESSIBLE UUID '26T']

(Note the similarity with 4.d.6).

E3.h.41 Prerequisites:3.h.34,3.b.36,3.d.12,3.i.6,3.h.40. [26V]

Let X be an ordinal, we define

$$x \le y \doteq x \in y \lor x = y$$

we know from 3.h.34 that $x \le y$ is a (possibly partial) order relation in X. Prove that $x \le y$ is a well order.

Hidden solution: [UNACCESSIBLE UUID '26W']

Remark 3.h.42. *Consider again Proposition 3.h.19 that states that $\mathbb{N}_{ZF}$ is well ordered by the relation $\subseteq$.* [275]

We know by 3.h.39 and 3.h.24 that $\mathbb{N}_{ZF}$ is an ordinal; we may be tempted to see Proposition 3.h.19 as a corollary of the previous result 3.h.41.

This is unfortunately not a well posed way of proving this result, because of this cascade of dependencies:

- *the proof of 3.h.41 relies on the result 3.d.12*
- *the result 3.d.12 in turn needs a definition by recurrence of a function: this is Theorem 4.b.1*
- *the proof of Theorem 4.b.1 uses the fact that the induction principle holds on $\mathbb{N}$.*

So we need to first prove the properties of $\mathbb{N}_{ZF}$ independently of the theory of ordinals, and then prove the results in Sec. §4, and then eventually we can prove the result 3.h.41, that states that any ordinal is well ordered by the relation $\subseteq$.

§3.i Well ordering [1YQ]

Definition 3.i.1. [07R]

*A total order $\le$ on a set X is a **well ordering** if every nonempty subset of X has a minimum.*

In particular X has a minimum that we will indicate with 0_X.

The theory of well orderings is very much linked to the theory of ordinals, of which we have given a few hints in Sec. §3.h.d. We just say that every ordinal is the standard representative of a type of well ordering. Using standard ordinal theory (due to Von Neumann) many of the subsequent exercises can be reformulated and simplified. [222]

Remark 3.i.2. *Recall that the supremum $\sup A$ of $A \subseteq X$ is (by definition) the minimum of the majorants (quando questo minimo esiste).* [07S] (Solved on 2023-01-17)

If X is well ordered we have the existence of the supremum $\sup A$ for any $A \subseteq X$ that is upper bounded. [†31] *(If A is not upper bounded, we can conventionally decide that $\sup A = \infty$).*

Exercises

E3.i.3 If $(X, \le)$ is a well-ordered set and $Y \subseteq X$ is a subset, then Y (with the restriction of the ordering) $\le$ is a well-ordered set. [07W]

E3.i.4 `Prerequisites:4.b.1`. Let X be totally ordered set. Show that these are equivalent: [07X] (Solved on 2023-01-17) (Proposed on 2022-12)

1. X is well ordered;
2. there are no strictly decreasing functions $f : \mathbb{N} \to X$.

(This is a special case of 3.d.12)

E3.i.5 `Prerequisites:3.d.58,3.d.34,3.d.51,3.i.4.Difficulty:*.Note:`*exercise 2 written exam on 29 January 2021.* [22F] (Solved on 2022-10-13 in part)

Let be given $(X, \le_X)$ where X is an infinite set and $\le_X$ is a well ordering.

- If X has no maximum, then there exists $(Y, \le_Y)$ such that setting $Z = Y \times \mathbb{N}$ with $\le_Z$ the lexicographical order, then $(X, \le_X)$ and $(Z, \le_Z)$ have the same type of order.
- If instead X has maximum, then there exist $(Y, \le_Y)$ and $k \in \mathbb{N}$ such that, setting Z be the concatenation of $Y \times \mathbb{N}$ and $\{0, \dots k\}$ (where $Y \times \mathbb{N}$ has the lexicographical order, as above), then $(X, \le_X)$ and $(Z, \le_Z)$ have the same type of order.
- Show that, in the previous cases, Y is well ordered.

Hidden solution: *[UNACCESSIBLE UUID '22G']*

E3.i.6 `Difficulty:*`. Let the ordered set $(X, \le)$ be given; we define [0DQ] (Proposed on 2022-12)

$$P_x \stackrel{\text{def}}{=} \{w \in X : w < x\} \quad .$$

Suppose $(X, \le)$ meets these two requirements:

[†31] "Upper bounded" means that there exists $w \in X$ such that $x \le w$ for every $x \in A$. This is equivalent to saying that the set of majorants of A is not empty!

- $$\forall x, y \in X\ ,\ P_x = P_y \Rightarrow x = y$$
- every non-empty set $A \subseteq X$ contains at least one minimal element, i.e.
$$\exists a \in A, \forall b \in A \neg(b < a) \quad ;$$

then $(X, \leq)$ is well ordered.

Hidden solution: [UNACCESSIBLE UUID '26R']

§3.i.a Successor

Definition 3.i.7. *Let X be a well-ordered non-empty set. Suppose $x \in X$ is not the maximum, then the set of majorants $\{y \in X : y > x\}$ is not empty, so we define the successor element $S(x)$ of x as* [1Z0]

$$S(x) = \min\{y \in X : y > x\} \quad .$$

Exercises

E3.i.8 Prerequisites:3.i.7. [1Z1]

(Proposed on 2023-01-17)

Suppose X has no maximum; let S be defined as in 3.i.7; show that is an injective function
$$S : X \to X \quad ,$$
and that $S(x) \neq 0_X$, for every x (that is, 0_X is not successor of any element).

Hidden solution: [UNACCESSIBLE UUID '223']

We note that in general S will not be surjective, as a function $S : X \to X \setminus \{0_X\}$: there may be elements $y \in S, y \neq 0_X$ that are not successors of an element. If, however, for a given $y \in X$, there exists $x \in X$ such that $y = S(x)$, we will say that x *is the* ***predecessor*** *of* y.

E3.i.9 Prerequisites:3.i.7,3.i.8.If $x \leq y \leq S(x)$ then $y = x \lor y = S(x)$. [22H]

Hidden solution: [UNACCESSIBLE UUID '22J']

(The meaning of this result is that $S(x)$ is the immediate successor of x, there is nothing in between...).

§3.i.b Segments and well orderings

In the following $(X, \leq_X)$ will be a well-ordered set.

Definition 3.i.10. *A nonempty subset $S \subseteq X$ is an* ***initial segment*** *if $\forall x \in S, \forall y \in X,\ y < x \Rightarrow y \in S$.* [07T]

Exercises

E3.i.11 Show that the initial segment union is an initial segment. [07Z]

E3.i.12 If $S \subseteq X$ is an initial segment and $S \neq X$, show that $s \in X \setminus S$ exists and is unique (s is called *the next item* to S) which extends S, *i.e.* such that $S \cup \{s\}$ is an initial segment. *Hidden solution:* *[UNACCESSIBLE UUID '081']* (Note that there are similarities with the concept of "successor" seen in 3.i.7... We could say that s is the successor of the segment S). [080] *(Solved on 2023-01-17)*

E3.i.13 `Prerequisites: 3.d.44, 3.d.45, 3.d.49`. Let X be a well-ordered set. Show that if $I \subseteq X$ is an interval then $I = [a, b)$ or $I = [a, b]$ or $I = [a, \infty)$ with $a, b \in X$. (The reverse is obviously true). [082] *(Solved on 2023-01-17)*

In particular, an initial segment is $[0_X, b)$ or $[0_X, b]$ or all X. *Hidden solution:* *[UNACCESSIBLE UUID '083']*

E3.i.14 Let $(X, \leq_X), (Y, \leq_Y)$ be totally ordered non-empty sets. Let $f : X \to Y$ be a strictly increasing bijective function. Then for each $S \subseteq X$ initial segment we have that $f(S)$ is an initial segment of Y; and vice versa. *Hidden solution:* *[UNACCESSIBLE UUID '085']* [084]

E3.i.15 `Prerequisites:3.i.4,4.b.1,3.d.51`. [086]

Let $(X, \leq_X)$ be a well-ordered non-empty set. Show that if $S \subseteq X$ is an initial segment and $(X, \leq_X)$ and $(S, \leq_X)$ are equiordinate from the map $f : S \to X$ then $X = S$ and f is the identity.

Hidden solution: *[UNACCESSIBLE UUID '087'][UNACCESSIBLE UUID '088']*

(Note the difference with cardinality theory: An infinite set is in one-to-one correspondence with some of its proper subsets, *cf* 3.j.36 and 3.j.39. Moreover, if two sets have the same cardinality then there are many bijections between them.)

E3.i.16 Give an example of a totally ordered set $(X, \leq_X)$ which has minimum, and of an initial segment S such that $(X, \leq_X)$ and $(S, \leq_X)$ are equiordinate. *Hidden solution:* *[UNACCESSIBLE UUID '08B']* [089] *(Solved on 2023-01-17)*

E3.i.17 `Prerequisites:3.i.15`. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be well ordered; suppose there exists a bijective function $f : X \to T$ strictly increasing where T an initial segment of Y; then f is unique (and unique is T). *Hidden solution:* *[UNACCESSIBLE UUID '08D']* [08C] *(Proposed on 2022-12)*

E3.i.18 `Prerequisites:3.e.24, 3.i.11, 3.i.14, 3.i.12, 3.i.17`. [08F]

Let be given two well-ordered non-empty sets $(X, \leq_X)$ and $(Y, \leq_Y)$. Show that

1. there is an initial segment S of X and a strictly increasing monotonic bijective function $g : S \to Y$; or [†32]
2. there is an initial segment T of Y and a bijective strictly increasing monotonic function $g : X \to T$.

In the first case we will write that $(Y, \leq_Y) \preceq (X, \leq_X)$, in the second that $(X, \leq_X) \preceq (Y, \leq_Y)$. (Note that in the first case you have $|Y| \leq |X|$ and in the second $|X| \leq |Y|$). By the previous exercise, the map g and its segment are unique.

Hidden solution: *[UNACCESSIBLE UUID '08G']*

E3.i.19 Prerequisites:3.i.18. Show that if $(X, \leq_X) \preceq (Y, \leq_Y)$ and also $(Y, \leq_Y) \preceq (X, \leq_X)$, then they are *equiordinate*. [08H]

Hidden solution: *[UNACCESSIBLE UUID '08J']*

The relation $\preceq$ is therefore a total order between types of well-orderings.

§3.i.c Examples

Exercises

E3.i.20 Prerequisites:3.d.37. The type of well ordering of $\mathbb{N}$ is called ω. Given $k \geq 2$ natural, $\mathbb{N}^k$ endowed with the lexicographical order is a well-ordered set (for 3.d.37), and the type of ordering is called ω^k. Show that $\omega^k \preceq \omega^h$ for $h > k$, and that ω^k, ω^h do not have the same type of order. [08K]

E3.i.21 Difficulty:*. Build a well ordering on a countable set X such that $X = \bigcup_{n=1}^{\infty} S_n$ where S_n are initial segments, each with order type ω^n. The order so built on X is indicated by ω^ω. *Hidden solution:* *[UNACCESSIBLE UUID '08N']* [08M]

E3.i.22 Difficulty:*. Build a strictly increasing map between ω^ω and $\mathbb{R}$. *Hidden solution:* *[UNACCESSIBLE UUID '08Q']* [08P]

§3.j Cardinality [1YW]

For convenience we will use the symbol $|A|$ to indicate cardinality of the set A. This symbol is used as follows. Given two sets A, B, we will write $|A| = |B|$ if these sets are **equipotents** (or sometimes **equinumerous**), *i.e.* if there is a bijective function between A and B; we will write $|A| \leq |B|$ if there is an injective function from A to B. We will also write $|A| < |B|$ if there is an injective function from A to B, but not a bijection.If we assume the axiom of choice to be true, then for every pair of sets we always have $|A| \leq |B|$ or $|B| \leq |A|$ (see 3.j.20). [22B]

Proposition 3.j.1. *If we now fix a family $\mathcal{F}$ of sets of interest, we first define the relation $A \sim B \iff |A| = |B|$ in it; it is easily shown that this is an equivalence relation; so we get that $|A| \leq |B|$ is a total order in $\mathcal{F}/\sim$.* [1Z9]

Proof. This derives from the Proposition 3.g.3, since the relation

$$ARB \iff |A| \leq |B|$$

is reflexive and transitive, and by Cantor–Bernstein's Theorem

$$|A| \leq |B| \wedge |B| \leq |A| \iff A \sim B \quad .$$

□

In the following, let $E_0 = \emptyset$, and let $E_n = \{1, \dots n\}$ otherwise if $n \geq 1$.

Lemma 3.j.2. *If $n, m \in \mathbb{N}, n < m$ then $|E_n| < |E_m|$. This is proven in Lemma 1.12.1 of the notes [2].* [2GK]

Definition 3.j.3. *By definition* [†33] "a set A is **finite** and has cardinality n" *if it is equipotent to a set E_n (for a choice of $n \in \mathbb{N}$; note that there is at most one n for which this may hold, by the above Lemma).So when the set is finite, $|A|$ is identified with the natural number of its elements; we will write $|A| = n$. If a set isn't finite, then it is **infinite**.* [1B1]

Note that the null map $f : \emptyset \to \emptyset$ is a bigection; and $|A| = 0 \Leftrightarrow A = \emptyset$. The following exercise is a fundamental result.

Exercise 3.j.4. *Prove that $\mathbb{N}$ is infinite, and that $|\mathbb{N}| > n, \forall n \in \mathbb{N}$.* Hidden solution: [UNACCESSIBLE UUID '2GJ'] [2GH]

We recall Theorem 1.12.2 of the notes [2], for convenience.

Theorem 3.j.5. *If A is infinite then $|A| \geq |\mathbb{N}|$. In particular, $|A| > n$ for any $n \in \mathbb{N}$.* [02S]

Definition 3.j.6. *A set A equipotent to $\mathbb{N}$ is called **countably infinite**;* [†34] *such a set is infinite (by the result 3.j.4 above).* [2DD]

§3.j.a Finite sets

Exercises

E3.j.7 If A is a finite set and $B \subseteq A$, prove that B is finite. [02T]

Hidden solution: *[UNACCESSIBLE UUID '02V']*

E3.j.8 Suppose we have a finite number $m \geq 1$ of sets $A_1, \ldots, A_m$ all finite. Show that $\bigcup_{j=1}^{m} A_j$ is a finite set. *Hidden solution:* *[UNACCESSIBLE UUID '02X']* [02W]

E3.j.9 Recall that A^B is the set of all functions $f : B \to A$. If A, B are finite non-empty sets show that $|A^B| = |A|^{|B|}$. What happens if one set, or both sets, are empty? *Hidden solution:* *[UNACCESSIBLE UUID '02Z']* [02Y]

E3.j.10 If A, B are finite non-empty sets, calculate the cardinality of the set of injective functions $f : B \to A$; and the cardinality of the surjective ones. What happens if one, or both, of the two sets A, B are empty? [22K]

§3.j.b Comparison

Exercises

E3.j.11 Prerequisites:3.e.20.Suppose A is not empty. We have $|A| \leq |B|$ if and only if there is a surgective function $f : B \to A$. (The "if" implication necessitates the axiom of choice; See also 3.b.46.) [030]

E3.j.12 Show that if $|A_1| \leq |A_2|$ and $|B_1| \leq |B_2|$ then $|A_1 \times B_1| \leq |A_2 \times B_2|$. [031]

E3.j.13 Show that if $|A_1| \leq |A_2|$ and $|B_1| \leq |B_2|$ then $|A_1^{B_1}| \leq |A_2^{B_2}|$ *Hidden solution:* *[UNACCESSIBLE UUID '033']* [032]

E3.j.14 Show that $|(A^B)^C| = |A^{(B \times C)}|$. *Hidden solution:* *[UNACCESSIBLE UUID '035']* [034] *(Solved on 2021-11-04)*

E3.j.15 Let I be a family of indices and B_i, A_i sets, for $i \in I$, such that $|A_i| \leq |B_i|$; suppose that the sets B_i are pairwise disjoint. Show that [036]

$$\left| \bigcup_{i \in I} A_i \right| \leq \left| \bigcup_{i \in I} B_i \right| .$$

(In your opinion, is it possible to prove this result without using the axiom of choice, at least in the case in which I is countable?) *Hidden solution:* [UNACCESSIBLE UUID '037']

E3.j.16 Let C be a set, I a family of indexes, and then B_i sets, for $i \in I$; suppose the sets B_i are pairwise disjoint; define $\mathcal{B} = \bigcup_{i \in I} B_i$ for convenience; then show that [038]

$$\forall i, |B_i| \leq |C| \quad \Rightarrow \quad |\mathcal{B}| \leq |I \times C| \tag{3.j.17}$$

$$\forall i, |B_i| \geq |C| \quad \Rightarrow \quad |\mathcal{B}| \geq |I \times C| \quad . \tag{3.j.18}$$

Hidden solution: [UNACCESSIBLE UUID '039']

E3.j.19 Let C be a set, I a family of indices, and B sets for $i \in I$ with $|B_i| = |C|$; then show that [03C]

$$|\mathcal{B}| = |C^I|$$

where $\mathcal{B} = \prod_{i \in I} B_i$. *Hidden solution:* [UNACCESSIBLE UUID '03D']

E3.j.20 `Prerequisites:3.e.24`. Show that cardinalities are always comparable: given two sets A, B either $|A| \leq |B|$ or $|B| \leq |A|$ holds. (Use Zorn's lemma and the construction explained in the exercise 3.e.24). *Hidden solution:* [UNACCESSIBLE UUID '03G'] [03F] *(Proposed on 2022-12)*

This statement is equivalent to the Axiom of Choice, see [21].

§3.j.c Countable cardinality

Definition 3.j.21. *Recall that a set is "countably infinite" if it has the same cardinality of* $\mathbb{N}$. [2DF]

If A is countably infinite, there exists $a : \mathbb{N} \to A$ bijective. Writing a_n instead of $a(n)$, we will therefore say that $A = \{a_0, a_1, a_2 \ldots\}$ is an ***enumeration****.*

Exercises

E3.j.22 Found a polynomial $p(x, y)$ which, seen as a function $p : \mathbb{N}^2 \to \mathbb{N}$ is bijective. [03H]
It follows, iterating, that there is a polynomial q_k in k variables $q_k : \mathbb{N}^k \to \mathbb{N}$ that is bijective. So $\mathbb{N}^k$ is countable. *Hidden solution:* [UNACCESSIBLE UUID '03J'] [UNACCESSIBLE UUID '03K']

E3.j.23 Show that the sets $\mathbb{Z}, \mathbb{Q}$ are countable. *Hidden solution:* [UNACCESSIBLE UUID '03N'] [03M]

E3.j.24 `Prerequisites: 3.j.15,3.j.22`. [03P]

†32 The two conditions can also both apply, in which case X, Y have the same type of order.

†33 This is the definition presented in the course. There are also other definitions of "finite set" [16]. See for example the exercise 3.j.39

†34 Attention, in English the term *countable* is used for finite or countable sets. By comparison, in Italian the term *insieme numerabile* is used to denote a *countably infinite* set.

Let $A_0,A_1 \ldots A_n \ldots$ sets of countable cardinality, for $n \in \mathbb{N}$.

Show that $B = \bigcup_{k=0}^{\infty} A_k$ is countable.

Note that B is infinite-countable if for example there is at least one n for which A_n is infinite-countable.

Hidden solution: *[UNACCESSIBLE UUID '03Q']*

E3.j.25 We indicate with $\mathcal{P}_f(A)$ the set of subsets $B \subseteq A$ which are finite sets. This is called colloquially *the set of finite parts*. [03R]

Show that $\mathcal{P}_f(\mathbb{N})$ is countably infinite.

Hidden solution: *[UNACCESSIBLE UUID '03S']*

This result applies in general, see 3.j.47.

§3.j.d Cardinality of the continuum

Definition 3.j.26. *We will say that a set has cardinality of the continuum if it has the same cardinality as* $\mathbb{R}$. [03V]

Remark 3.j.27. *Cantor proved that* $|\mathbb{N}| < |\mathbb{R}|$. *Cantor then (in 1878) formulated the* continuum hypothesis *CH:* for any infinite set $E \subseteq \mathbb{R}$, either $|E| = |\mathbb{R}|$ or $|E| = |\mathbb{N}|$. *For many year mathematicians tried to prove (or disprove) CH. It took decades to understand that this was not possible. We know know that, if ZF is consistent, then neither CH nor its negation can be proven as theorems in ZF (even using the Axiom of Choice). The second part of the statement was proved by Gödel nel 1939. The first part by Cohen in 1963. See Chap. 6 in [12].* [2F2]

Exercises

E3.j.28 Explain how you could explicitly construct a bijection between $[0,1)$ and $[0,1)^2$. [03X] *(Proposed on 2022-12)*

E3.j.29 Show with explicit constructions that the following sets have continuum cardinalities: [03Y] *(Proposed on 2022-10-13)* *(Solved on 2022-10-27)*

$$[0,1], \quad [0,1), \quad (0,1), \quad (0,\infty) \ .$$

Hidden solution: *[UNACCESSIBLE UUID '03Z']*

E3.j.30 `Prerequisites:3.j.37, 3.j.14, 3.j.13.` [040]

Show that the following sets have continuum cardinalities.

$$\mathbb{R}^n, \quad \{0,1\}^{\mathbb{N}}, \quad \mathbb{N}^{\mathbb{N}}, \quad \mathbb{R}^{\mathbb{N}} \ .$$

Hidden solution: *[UNACCESSIBLE UUID '041'][UNACCESSIBLE UUID '042']*

E3.j.31 `Prerequisites:3.j.15,3.j.30.` Let A_t be sets with cardinality less than or equal to the continuum, for $t \in \mathbb{R}$. Show that $\bigcup_{t\in\mathbb{R}} A_t$ has cardinality of the continuum. *Hidden solution:* *[UNACCESSIBLE UUID '044']* [043]

E3.j.32 Let $\mathcal{A}$ be the set of subsets $B \subseteq \mathbb{R}$ which are countable sets; show that $\mathcal{A}$ has cardinality of the continuum. *Hidden solution:* *[UNACCESSIBLE UUID '046']* [045]

§3.j.e In general

Let's add some more general exercises.

Exercises

E3.j.33 Show that $|A| < |\mathcal{P}(A)|$. *Hidden solution:* *[UNACCESSIBLE UUID '049']* [048]

E3.j.34 Consider the set $\mathbb{N}^\mathbb{N}$ of functions $f : \mathbb{N} \to \mathbb{N}$; and the subset $\mathcal{A}$ of f of functions that can be defined using an algorithm, written in a programming language of your choice (also assuming that the computer that is running this algorithm has potentially unlimited memory) and such that for each choice $n \in \mathbb{N}$ in input the algorithm must finish and return $f(n)$. Compare the cardinalities of $\mathbb{N}^\mathbb{N}$ and $\mathcal{A}$. [04B]

E3.j.35 Calculate the cardinality of the set $\mathcal{F}$ of weakly decreasing functions $f : \mathbb{N} \to \mathbb{N}$. *Hidden solution:* *[UNACCESSIBLE UUID '04F']* [04D] *(Proposed on 2022-12)*

E3.j.36 `Prerequisites:4.b.1.` [04G]

A set A is called *Dedekind–infinite* if A is in bijection with a proper subset, that is if there is $B \subset A, B \neq A$ and $h : A \to B$ bijection. Show that a set A is *Dedekind–infinite* if and only if there is an injective function $g : \mathbb{N} \to A$. (This result does not require the axiom of choice.)

Hidden solution: *[UNACCESSIBLE UUID '04H']*

E3.j.37 `Prerequisites:3.j.5.` If A is infinite and B is countable, show that $|A| = |A \cup B|$ using the existence of an injective function $g : \mathbb{N} \to A$. [04J]

Hidden solution: *[UNACCESSIBLE UUID '04K']*

E3.j.38 `Prerequisites:3.j.8,3.j.37.` Similarly if A is infinite and B is finite show that $|A| = |A \setminus B|$ using the fact that for every infinite set X there is an injective $g : \mathbb{N} \to X$ function. *Hidden solution:* *[UNACCESSIBLE UUID '22N']* [22M]

E3.j.39 `Prerequisites:3.j.5, 3.j.36.` Show that a set A is *Dedekind–infinity* if and only if it is infinite (according to the definition seen at the beginning of the chapter). [04M]

Hidden solution: *[UNACCESSIBLE UUID '04N']*

Note: According to [10], the previous equivalence cannot be proved using only the axioms of ZF (Zermelo–Fraenkel without the axiom of choice) ; the previous equivalence can be proved using the axioms of ZFC (Zermelo–Fraenkel with the axiom of choice); but its validity in ZF is weaker than the axiom of choice.

E3.j.40 `Prerequisites:3.j.37,3.e.24.Difficulty:*.` [04P] *(Proposed on 2022-12)*

Let X an infinite set. Show that X can be partitioned in two sets X_1, X_2 that have the same cardinality as X. *(Hint. consider subsets of X on which the property is valid, use Zorn) Hidden solution:* *[UNACCESSIBLE UUID '04Q']*

E3.j.41 `Prerequisites:3.j.40,3.j.22,3.i.5.Difficulty:*.` [04R] *(Solved on 2022-10-13 in parte)*

Let A infinite. Show that $|D \times A| = |A|$ for every non-empty countable set D. [†35]

(A possible solution uses 3.j.40) *Hidden solution:* *[UNACCESSIBLE UUID '04S']*

(Another possible solution uses Zermelo's theorem, 3.i.5 and 3.j.22; in this case 3.j.40 becomes a corollary of this result.) *Hidden solution:* *[UNACCESSIBLE UUID '04T']*

[†35] Equivalently, show that there is a partition U of A such that each part $B \in U$ has cardinality $|B| = |A|$, and the family U of the parts has cardinality $|U| = |D|$.

E3.j.42 Let A, B be infinite. Show that $|A \cup B| = \max\{|A|, |B|\}$. *Hidden solution:* *[UNACCESSIBLE UUID '04W']* [04V] (Solved on 2022-10-27)

E3.j.43 Show that if A is an infinite set, and decomposes into the disjoint union of two sets A_1, A_2 with $|A_1| \le |A_2|$ then $|A| = |A_2|$. *Hidden solution:* *[UNACCESSIBLE UUID '04Y']* [04X]

E3.j.44 Prerequisites:3.e.24, 3.j.41.Difficulty:**. [04Z] (Proposed on 2022-10-13) (Solved on 2022-11-15)
Let A, B be infinite sets. Show that $|A^2| = |A|$.

Use this result to show that if A, B are not empty and at least one is infinite then $|A \times B| = \max\{|A|, |B|\}$.

Hidden solution: *[UNACCESSIBLE UUID '050']*

See also the note 3.j.45.

Remark 3.j.45. *Historical notes.* [27H]

- *The proposition* "$|A^2| = |A|$ holds for every infinite set" *is equivalent to the axiom of choice. This was demonstrated by Tarski [23] in 1928 ; the article is online and downloadable and contains other surprising equivalences. See also [21] Part 1 Section 7 page 140 assertion CN6.*

- *Jan Mycielski [18] reports:* «Tarski told me the following story. He tried to publish his theorem (stated above) in the *Comptes Rendus Acad. Sci. Paris* but Fréchet and Lebesgue refused to present it. Fréchet wrote that an implication between two well known propositions is not a new result. Lebesgue wrote that an implication between two false propositions is of no interest. And Tarski said that after this misadventure he never tried to publish in the Comptes Rendus».

  This anecdote shows how in the past (before the works of Godel and Cohen [5], even the most respected mathematician had a feeble grasp of the importance of the Axiom of Choice.

Exercises

E3.j.46 Prerequisites:3.j.44.Let A be an infinite set. Let $n \in \mathbb{N}$ with $n \ge 1$. Show that $|A^n| = |A|$. *Hidden solution:* *[UNACCESSIBLE UUID '052']* [051]

E3.j.47 Prerequisites:3.j.15, 3.j.46, 3.j.44. Let A be an infinite set. Show that the set of finite parts $\mathcal{P}_f(A)$ has the same cardinality as A. *Hidden solution:* *[UNACCESSIBLE UUID '054']* [053] (Proposed on 2022-12) (Solved on 2023-01-24)

E3.j.48 Prerequisites: 3.j.16, 3.j.44. Let X be an infinite set, let $\sim$ be an equivalence relation on X, let $U = X/\sim$ be the equivalence classes. [055] (Solved on 2023-01-24)

- Suppose each class is finite, show that $|U| = |X|$.
- Suppose U is infinite and every class has cardinality at most $|U|$, then $|U| = |X|$.

Hidden solution: *[UNACCESSIBLE UUID '056']*

E3.j.49 Prerequisites:3.b.47,3.j.47,3.j.48.Difficulty:**. [057]

Let V be a real vector space. Let A, B be two Hamel bases (see 3.b.47). Show that $|A| = |B|$. (This result is known as "*Dimension theorem*")

More in general, let $L, G \subseteq V$, if the vectors in L are linearly independent, and G generates V, show that $|L| \le |G|$.

Hidden solution: [UNACCESSIBLE UUID '058']

Other interesting exercises are 10.g.10, 10.a.7.

§3.j.f Power

Recall that A^B is the set of all functions $f : B \to A$. We will write $|2^A|$ to indicate the cardinality of the set of parts of A.

Exercises

E3.j.50 Prerequisites:3.j.44. Let A, B be non-empty sets and such that A is infinite and $2 \le |B| \le |A|$ then $|B^A| = |2^A|$. *Hidden solution:* [UNACCESSIBLE UUID '05K'] [05J] (Proposed on 2022-12)

E3.j.51 Let A, B be non-empty sets, suppose there is a C such that $|B| = |2^C|$ then $|B^A| = \max\{|B|, |2^A|\}$. [05M]

Hidden solution: [UNACCESSIBLE UUID '05N']

In general in case $|B| > |A|$ the study of the cardinality of $|B^A|$ is very complex (even in seemingly simple cases like $A = \mathbb{N}$).

§3.k Operations on sets [1YX]

Exercises

E3.k.1 Let X be a non-empty set, and $A \subseteq X$. We will denote with $A^c = X \setminus A = \{x \in X : x \notin A\}$ the complement of A in X. [05R]

We define the characteristic function $\mathbb{1}_A : X \to \mathbb{Z}$ by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

Prove that

$$\mathbb{1}_{A^c} = 1 - \mathbb{1}_A \ , \ \mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B \ , \ \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

E3.k.2 Now consider instead the characteristic function defined as before, but considered as $\mathbb{1}_A : X \to \mathbb{Z}_2$ i.e. taking values in the group $\mathbb{Z}_2$ (more correctly referred to as $\mathbb{Z}/2\mathbb{Z}$). [05S]

In this case the above relations can be written as

$$\mathbb{1}_{A^c} = \mathbb{1}_A + 1 \ , \ \mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B \ , \ \mathbb{1}_{A\cup B} = \mathbb{1}_A \mathbb{1}_B + \mathbb{1}_A + \mathbb{1}_B .$$

Recall the definition of the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A)$, and then

$$\mathbb{1}_{A\Delta B} = \mathbb{1}_A + \mathbb{1}_B .$$

With these rules we show that

$$A\Delta B = B\Delta A \ , \ (A\Delta B)^c = A\Delta(B^c) = (A^c)\Delta B \ , \ A\Delta B = C \iff A = B\Delta C$$

$$(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C) \ , \ A \cup (B\Delta C) = (A \cup B)\Delta(A^c \cap C)$$

E3.k.3 Let A, B, C sets, then [05T]

$$A \times (B \cup C) = (A \times B) \cup (A \times C) \ ,$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C) \ .$$

Therefore the Cartesian product operation is distributive on the union and intersection.

E3.k.4 If A, B, C are non-empty sets and [05V]

$$(A \times B) \cup (B \times A) = (C \times C)$$

then $A = B = C$.

Hidden solution: *[UNACCESSIBLE UUID '05W']*

E3.k.5 Given four sets X, Y, A, B with $A \subset X, B \subset Y$, write [05X]

$$(X \times Y) \setminus (A \times B)$$

as a union of three sets, pairwise disjoint, each a Cartesian product.

Hidden solution: *[UNACCESSIBLE UUID '05Y']*

E3.k.6 We want to rewrite the tautologies seen in 3.a.10 in the form of set relations. [05Z]

Let X be a set and let $\alpha, \beta, \gamma \subseteq X$ be subsets. Let $x \in X$. If we define $A = (x \in \alpha)$, $B = (x \in \beta)$, $C = (x \in \gamma)$ in the tautologies, we can then rewrite each tautology as a formula between sets $\alpha, \beta, \gamma, X, \emptyset$, that use connectives $=, \cap, \cup$ and the complement.

Surprisingly, rewriting can be done algorithmically and in a purely syntactic manner. Pick a tautology seen in 3.a.10. In the following φ, ψ indicate subparts of tautology that are well-formed formulas.

- Replace $((\varphi) \Rightarrow (\psi))$ with $((\neg(\varphi)) \vee (\psi))$ (you will get another tautology).
- Then syntactically replace $\neg(\varphi)$ with $(\varphi)^c$, $\vee$ with $\cup$ and $\wedge$ with $\cap$; replace A with α, B with β, C with γ, V with X, and F with $\emptyset$.
- Finally, if the formula contains at least one " $\iff$ ", transform them all in "$=$"; otherwise add "$= X$" at the end.

Check that this "algorithm" really works!

E3.k.7 Let X be a set. Let I, J families not empty of indexes, and for every $i \in I$ let $J_i \subseteq J$ a family not empty of indexes. For each $i \in I, j \in I_j$ let $A_{i,j} \subseteq X$. Show that [060]

$$\bigcap_{i\in I} \bigcup_{j \in J_i} A_{i,j} = \bigcup_{\beta \in B} \bigcap_{i \in I} A_{i,\beta(i)}$$

where $B = \prod_{i\in I} J_i$ and remember that every $\beta \in B$ is a function $\beta : I \to J$ for which for every i you have $\beta(i) \in J_i$. Then formulate a similar rule by exchanging the role of intersection and union. (use the complements of the sets $A_{i,j}$ and the rules of de Morgan). *Hidden solution:* *[UNACCESSIBLE UUID '061']*

§3.k.a Limsup and liminf of sets

Definition 3.k.8. *Given $A_1, A_2 \dots$ sets , for $n \in \mathbb{N}$, we define* [1Z2] *(Solved on 2022-11-29)*

$$\limsup_{n\to\infty} A_n \stackrel{def}{=} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \tag{3.k.9}$$

$$\liminf_{n\to\infty} A_n \stackrel{def}{=} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \tag{3.k.10}$$

(3.k.11)

We suppose that $A_n \subseteq X$ for every n. (We can set $X = \bigcup_n A_n$).

Exercises

E3.k.12 Recall that [063]

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

is the complement of A in X (as defined in 3.b.6). Show that

$$(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty}(A_n^c) \ .$$

E3.k.13 Prerequisites:4.g.1. Show that [064]

$$\limsup_{n\to\infty} A_n = \{x \in X : x \in A_n \text{ frequently in } n\} \ , \tag{3.k.14}$$

$$\liminf_{n\to\infty} A_n = \{x \in X : x \in A_n \text{ eventually in } n\} \ . \tag{3.k.15}$$

(”Frequently” and ”eventually” are discussed in Sec. §4.g).

E3.k.16 Prerequisites:3.k.13, 4.g.6. Given sets $A_1, A_2 \dots$ and $B_1, B_2 \dots$, for $n \in \mathbb{N}$, say if there is a relation (of equality or containment) between [065]

$$(\liminf_{n\to\infty} A_n) \cap (\liminf_{n\to\infty} B_n) \stackrel{?}{=} \liminf_{n\to\infty}(A_n \cap B_n) \ , \tag{3.k.17}$$

$$(\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n) \stackrel{?}{=} \liminf_{n\to\infty}(A_n \cup B_n) \ . \tag{3.k.18}$$

If equality does not hold, show an example. Then use 3.k.12 to establish similar rules for $\limsup_{n\to\infty} A_n$.

Hidden solution: [UNACCESSIBLE UUID '066']

§3.l Combinatorics

Exercises

E3.l.1 Let be given n, k natural with $k \geq 1$. How many different choices of vectors $(j_1, \dots j_k)$ of natural numbers are there such that $j_1 + \cdots + j_k = n$? How many different choices of vectors $(j_1, \dots j_k)$ of positive natural numbers are there such that $j_1 + \cdots + j_k = n$? *Hidden solution:* [UNACCESSIBLE UUID '09P'] [09N]

E3.l.2 Let n, m be positive integers and let $I = \{1, \dots, n\}, J = \{1, \dots m\}$. [09Q]

- How many functions $f : I \to J$ are there?
- How many functions $f : I \to J$ are injective?
- How many functions $f : I \to J$ are strictly growing?
- How many functions $f : I \to J$ are weakly increasing?

Hidden solution: *[UNACCESSIBLE UUID 'O9R']*

See also exercise 3.j.9.

§4 Natural numbers [1X9]

We want to properly define the set

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

of the natural numbers.

A possible model, as shown in Sec. §3.h, is obtained by relying on the theory of Zermelo—Fraenkel.

Here instead we present Peano's axioms, expressed using the *naive version* of set theory.

Definition 4.1 (Peano's axioms). [1XB] (Solved on 2022-11-03)

(N1) *There is a number* $0 \in \mathbb{N}$.

(N2) *There is a function* $S : \mathbb{N} \to \mathbb{N}$ *(called "successor"), such that* [†36]

(N3) $S(x) \neq 0$ *for each* $x \in \mathbb{N}$ *and*

(N4) S *is injective, that is,* $x \neq y$ *implies* $S(x) \neq S(y)$.

(N5) *If* U *is a subset of* $\mathbb{N}$ *such that:* $0 \in U$ *and* $\forall x, x \in U \Rightarrow S(x) \in U$ *, then* $U = \mathbb{N}$.

We will often write Sn *instead* $S(n)$ *to ease notations.*

From those two important properties immediately follow. One is the principle of induction, see 4.a.1. The other is left for exercise.

Exercise 4.2. *Show that every* $n \in \mathbb{N}$ *with* $n \neq 0$ *is successor of another* $k \in \mathbb{N}$*, proving by induction on* n *this proposition* [1YP]

$$P(n) \stackrel{def}{=} (n = 0) \vee (\exists k \in \mathbb{N}, S(k) = n) \quad .$$

This shows that the successor function

$$S : \mathbb{N} \to \mathbb{N} \setminus \{0\}$$

is bijective.

If $n \neq 0$*, we will call* $S^{-1}(n)$ *the* ***predecessor*** *of* n.

Hidden solution: [UNACCESSIBLE UUID '22Q']

(Part of this result applies more generally, see 3.i.8)

The idea is that the successor function encodes the usual numbers according to the scheme

$$1 = S(0), \quad 2 = S(1), \quad 3 = S(2) \dots$$

and (having defined the addition) we will have that $S(n) = n + 1$.

Exercise 4.3. Prerequisites: 4.3. *Removing one of the axioms (N1)—(N5), describe a set that satisfies the others but it is not isomorphic to natural numbers.* [1XD]

Hidden solution: [UNACCESSIBLE UUID '22V']

§4.a Induction [27J]

Proposition 4.a.1 (Induction Principle). *Let $A \supseteq \mathbb{N}$ and $P(n)$ be a logical proposition that can be evaluated for $n \in A$. Suppose the following two assumptions are satisfied:* [1XC]

- *$P(n)$ is true for $n = 0$ and*
- *$\forall n \in \mathbb{N}, P(n) \Rightarrow P(S(n))$;*

then P is true for every $n \in \mathbb{N}$.

Proof. Let $U = \{n \in \mathbb{N} : P(n)\}$, we know that $0 \in U$ and that $\forall x, x \in U \Rightarrow S(x) \in U$, then from (N5) we conclude that $U = \mathbb{N}$. □

The verification of $P(0)$ is called the "basis of induction", while the verification of $\forall n \in \mathbb{N}, P(n) \Rightarrow P(S(n))$ it is called "inductive step" (in which $P(n)$ is taken as a hypothesis, and is called "inductive hypothesis").

Exercises

E4.a.2 Prove that $\forall n \in \mathbb{N}, n \neq S(n)$. [1XF]

Hidden solution: *[UNACCESSIBLE UUID '1XJ']*

E4.a.3 Prove [†37] by induction the following assertions: [1XG]

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$;
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$;
3. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$;
4. $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$;
5. $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$;
6. $n! \geq 2^{n-1}$;
7. If $x > -1$ is a real number and $n \in \mathbb{N}$ then $(1 + x)^n \geq 1 + nx$ (Bernoulli inequality).

Hidden solution: *[UNACCESSIBLE UUID '1XK']*

§4.b Recursive definitions [274]

Theorem 4.b.1. [08Z]

Let A be a non-empty set; suppose that $a \in A$ is fixed, and functions $g_n : A \to A$ are given, one for each $n \in \mathbb{N}$. Then there exists an unique function $f : \mathbb{N} \to A$ such that

- *$f(0) = a$, and*

[†36] We are using the same word *successor* used in the definition 3.i.7 for well ordered sets, and in 3.h.1 in Zermelo-Fraenkel theory: we will see how these definition are "compatible".

[†37] In the following exercises we give for good knowledge of the operations typical of the natural numbers, and their order relation.

- *for every $n \in \mathbb{N}$ we have $f(S(n)) = g_n(f(n))$.*

We will say that the function f is ***defined by recurrence*** *by the two previous conditions.*

Proof. Trace of proof. Note that the proof only uses Peano's axioms and induction. [090] Given $m \in \mathbb{N}, m \neq 0$ we recall that $S^{-1}(m)$ is the predecessor, see 4.2 (using the arithmetic we may write

$$S^{-1}(m) = m - 1 \ , \ \ S(k) = k + 1$$

but this theorem is needed to define the arithmetic...) For any given $R \subseteq \mathbb{N} \times A$ we define the projection on the first component

$$\pi(R) = \{n \in \mathbb{N}, \exists x \in A, (n, x) \in R\} \ .$$

Consider the family $\mathcal{F}$ of relations $R \subseteq \mathbb{N} \times A$ that satisfy

$$(0, a) \in R \tag{*}$$

$$\forall n \geq 0, \forall y \in A, \ \ (n, y) \in R \Rightarrow (S(n), g_n(y)) \in R \tag{**}$$

We show that under these conditions $\pi(R) = \mathbb{N}$; we know that $0 \in \pi(R)$; if $m \in \pi(R)$, then there exists $x \in A$ for which $(m, x) \in R$ from which for ** follows $(S(m), g_m(x)) \in R$, and we obtain $S(m) \in \pi(R)$.

The family $\mathcal{F}$ is not empty because $\mathbb{N} \times A \in \mathcal{F}$. Let then T be the intersection of all relations in $\mathcal{F}$. T is therefore the least relation in $\mathcal{F}$.

It is possible to verify that T satisfies the previous * and ** properties. In particular $\pi(T) = \mathbb{N}$.

We must now show that T is the graph of a function (which is the desired f function), that is, that for every n there is a single $x \in A$ for which $(n, x) \in T$.

Let $A_n = \{x \in A, (n, x) \in T\}$; we write $|A_n|$ to denote the number of elements in A_n; since $\pi(T) = \mathbb{N}$ then $|A_n| \geq 1$ for every n. We will show that $|A_n| = 1$ for each n. We will prove it by induction. Let

$$P(n) \doteq |A_n| = 1 \quad .$$

Let's see the induction step.

Suppose by contradiction that $|A_m| = 1$ but $|A_{Sm}| \geq 2$; morally at m the graph of the function f "forks" and the function becomes "multivalued". We define for convenience $w = g_m(x), k = Sm$; we may remove some elements to T (those that do not have a "predecessor" according to the relation **) defining

$$\tilde{T} = T \setminus \{(k, y) \, : \, y \in A, y \neq w\}$$

it is possible to show that $\tilde{T}$ satisfies * and **, but $\tilde{T}$ would be smaller than T, against the minimality of T. To prove that $P(0)$ holds, we define $k = 0, w = a$ and proceed in the same way.

The previous reasoning also shows that the function is unique, because if the graph G of any function satisfying to * and ** must contain T, then $T = G$. □

More generally given $g_n : A^{n+1} \to A$, an unique function $f : \mathbb{N} \to A$ exists, such that $f(0) = a$ and for every $n \in \mathbb{N}$ $f(S(n)) = g_n(f(0), f(1), \ldots f(n))$.

Exercises

E4.b.2 Prerequisites:4.b.1.Adapt the exercise 4.b.1 to define the *Fibonacci sequence*, which satisfies the rule [1X7]

$$a_0 = 1 \quad , \quad a_1 = 1 \quad , \quad a_n = a_{n-1} + a_{n-2}$$

for $n \geq 2$.

Hint. You don't have to rewrite the whole proof of 4.b.1, rather choose $A = \mathbb{N}^2$ and choose g with cunning.

Hidden solution: [UNACCESSIBLE UUID '1X8']

E4.b.3 Define the interval [294]

$$I_n = \{0, \dots n\}$$

of natural numbers using a recursive definition (without using the order relation).

Hidden solution: [UNACCESSIBLE UUID '295']

§4.c Arithmetic [0NN]

We will define the addition operation between natural numbers, formally

$$\cdot + \cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \quad , \quad (h, k) \mapsto h + k \quad .$$

Definition 4.c.1. *Having fixed the parameter $h \in \mathbb{N}$, we define the operation $h + \cdot$, which will be a function $f_h : \mathbb{N} \to \mathbb{N}$ given by $f_h(n) = h + n$, using a recursive definition: we wish to express the rules* [292]

- $h + 0 = h \quad ,$
- $\forall n \in \mathbb{N}, h + S(n) = S(h + n) \quad .$

To this end, set $A = \mathbb{N}$, and $g(n, a) = S(a)$, we rewrite the above as recursive rules for f_h

- $f_h(0) = h \quad ,$
- $\forall n \in \mathbb{N}, f_h(S(n)) = g(n, f_h(n)) = S(f_h(n)) \quad .$

This defines recursively f_h. Considering then the parameter h as a variable, we have constructed the addition operation, and we define the operation "+" between natural numbers as $h + n = f_h(n)$.

This operation is commutative and associative, as shown below.

Note that $h + 0 = f_h(0) = h$ (basis of recursion); also $0 + n = f_0(n) = n$ (shows easily by induction).

To prove that it is commutative, we first show that

Lemma 4.c.2. $\forall n, h \in \mathbb{N}, f_{S(h)}(n) = S(f_h(n))$ [27N]

(Solved on 2022-11-03)

Proof. Recall that $S(f_h(n)) = f_h(S(n))$ by recursive definition; Consider

$$P(n) \doteq \forall h \in \mathbb{N}, f_{S(h)}(n) = S(f_h(n)) \quad ;$$

$P(0)$ is the clause

$$\forall h \in \mathbb{N}, f_{S(h)}(0) = S(f_h(0)) = S(h)$$

which is true because it is the initial value of the recursive definition $f_{S(h)}$ and f_h. For the inductive step we assume that $P(n)$ is true and we study $P(S(n))$, within which we can say

$$f_{S(h)}(S(n)) \overset{(1)}{=} S(f_{S(h)}(n)) \overset{(2)}{=}$$
$$SS(f_h(n)) \overset{(3)}{=} S(f_h(S(n))$$

where in (1) we used the recursive definition of f_h with $S(h)$ instead h, in (2) we used the inductive hypothesis, and in (3) we used the recursive definition of f_h. This completes the inductive step.

(Note, in the first step, how important it is that in the definition of $P(n)$ there is $\forall h \in \mathbb{N}, \ldots$). □

Proposition 4.c.3. *(Replaces 27Y) Addition is commutative.* [27P]

Proof. By the lemma we can write

$$S(h) + n = S(h + n) = h + S(n) \tag{4.c.4}$$

intuitively the formula is symmetric and therefore also the definition of addition must have a symmetry. Precisely, let $\tilde{f}_n(h) \overset{\text{def}}{=} f_h(n)$ then $\tilde{f}_n(0) = n$ (as already noted) and for the lemma 4.c.2 $\tilde{f}_n(S(h)) = S\left(\tilde{f}_n(h)\right)$ but then $\tilde{f}$ satisfies the same recursive relation as f and therefore they are identical, so $f_h(n) = f_n(h)$. (The idea is that if we had defined addition. recursively starting from left instead of right, we would have achieved the same result). □

At this point we can give a name to $1 = S(0)$ and notice that $S(n) = n + 1$. So from now on we could do without the symbol S.

With similar procedures we show that addition is associative.

Proposition 4.c.5. *Addition is associative.* [27Q]

Proof. Consider

$$P(h) \doteq \forall n, m \in \mathbb{N}, (n + m) + h = n + (m + h) \quad ;$$

Obviously $P(0)$ is true, moreover $P(Sh)$ is proven (omitting "$\forall n, m \in \mathbb{N}$") like this

$$(n + m) + Sh = S(n + m) + h = (Sn + m) + h \overset{P(n)}{=}$$
$$= Sn + (m + h) = n + S(m + h) = n + (m + Sh) \,\square$$

Multiplication is similarly defined.

Definition 4.c.6. *We fix the parameter m, and we define recursively $(m \times \cdot)$ through* [28V] *(Solved on 2022-11-03)*

- $m \times 0 = 0$
- $\forall n \in \mathbb{N}, m \times (n + 1) = m \times n + m;$

then we can prove the known properties (commutativity, associativity, distributivity).

Exercises

E4.c.7 Rewrite some notions seen above, such as the principle of induction, and the definition of addition, using $n+1$ instead of $S(n)$. [27R]

E4.c.8 Show that the function $f_h(n) = (h+n)$ is injective. *Hidden solution:* [UNACCESSIBLE UUID '27T'] [27S] *(Solved on 2022-11-03)*

E4.c.9 Prove the *cancellation property*: if $n+h = m+h$ then $n = m$. [27V]

Hidden solution: [UNACCESSIBLE UUID '286']

E4.c.10 We have $n + m = 0$ if and only if $n = 0 \wedge m = 0$. *Hidden solution:* [UNACCESSIBLE UUID '285'] [27W] *(Solved on 2022-11-03)*

E4.c.11 You have $n \times m = 0$ if and only if $n = 0 \vee m = 0$. *Hidden solution:* [UNACCESSIBLE UUID '284'] [27X] *(Solved on 2022-11-03)*

E4.c.12 Prove that multiplication is commutative. Hint prove by induction in n [28T]

$$\forall m, n \in \mathbb{N}, (m+1) \times n = m \times n + n \quad ,$$

then reason as in Prop. 4.c.3. *Hidden solution:* [UNACCESSIBLE UUID '28W']

E4.c.13 Show that addition distributes over multiplication. Hint prove by induction in h [281]

$$\forall m, n, h \in \mathbb{N},\ m \times (n+h) = m \times n + m \times h \quad .$$

Hidden solution: [UNACCESSIBLE UUID '28Y']

E4.c.14 Prerequisites:4.c.13.Show that multiplication is associative. Hint prove by induction in h [27Z]

$$\forall m, n, h \in \mathbb{N}, (m \times n) \times h = m \times (n \times h) \quad .$$

Hidden solution: [UNACCESSIBLE UUID '28X']

E4.c.15 Fix $n \neq 0$ and $h \in \mathbb{N}$, write a recursive definition of *exponentiation* n^h. Then prove that $n^{h+k} = n^n n^k$ and $(n^h)^k = n^{(hk)}$. [280]

Hidden solution: [UNACCESSIBLE UUID '2DG']

In the following we will simply write nm instead of $n \times m$.

§4.d Ordering [27K]

Hypothesis 4.d.1. *We will study an order relation $\le$ on $\mathbb{N}$ (not necessarily total) such that* [26H]

$$\forall x \in \mathbb{N}, (0 \le x) \quad , \tag{4.d.2}$$

$$\forall x, y \in \mathbb{N}, (x < Sy) \iff (x \le y) \quad ; \tag{4.d.3}$$

where as usual

$$x < y \doteq (x \le y) \wedge (x \neq y) \quad .$$

Theorem 4.d.4. *There is an unique order relation $\le$ on $\mathbb{N}$ such that (4.d.3),(4.d.2) in 4.d.1 hold, and this ordering is well-ordered.* [26Y]

This theorem will be proven in the following: uniqueness in 4.d.5, well ordering in 4.f.6. The existence of such ordering is justified by the model in Z-F, as seen before and summarized in Section §4.e; otherwise the ordering can be defined using arithmetic, as shown in Section §4.d.a.

Exercises

E4.d.5 Suppose that $\preceq$ is a (possibly partial) order relation on $\mathbb{N}$ satisfying (4.d.3),(4.d.2) in 4.d.1 then $\preceq$ is unique. *Hidden solution:* [UNACCESSIBLE UUID '270'] [267]

E4.d.6 Let [271]

$$P_x = \{z \in \mathbb{N}, z < x\}$$

show that

$$\forall x, y \in \mathbb{N}\ ,\ P_x = P_y \Rightarrow x = y$$

using the properties in Hypothesis 4.d.1.

Hidden solution: [UNACCESSIBLE UUID '272']

(Note the similarity with 3.h.40).

E4.d.7 By setting $n = x = y$ in (4.d.3) we obtain that $n < Sn$. [276]

E4.d.8 Prerequisites:4.d.7,4.d.1.Using the properties in 4.d.1 and assuming that $\preceq$ is a total order relation (as will be proven), prove that [277]

$$\forall n, m \in \mathbb{N}, (n < m) \Rightarrow (Sn < Sm) \quad .$$

Hidden solution: [UNACCESSIBLE UUID '278']

E4.d.9 If $\preceq$ is a total order on $\mathbb{N}$ then these are equivalent [26X]

$$\forall x, y \in \mathbb{N}, (x \preceq y \preceq Sx) \Rightarrow (x = y \lor y = Sx) \quad , \tag{4.d.10}$$

$$\forall x, y \in \mathbb{N}, (x \prec Sy) \iff (x \preceq y) \quad ; \qquad \text{(as in (4.d.3))}$$

$$\forall x, y \in \mathbb{N}, (x \prec y) \iff (Sx \preceq y) \quad . \tag{4.d.11}$$

Note the analogy with 3.h.5

Hidden solution: [UNACCESSIBLE UUID '296']

§4.d.a Ordering from arithmetic [287]

Having already defined arithmetic, a convenient definition of ordering is as follows.

Definition 4.d.12. *Given $n, m \in \mathbb{N}$, we will say that $n \le m$ if there exists $k \in \mathbb{N}$ such that $n + k = m$* [288]

We will show that $\le$ it is a total order relation, and is a well ordering. Let's first see some elementary but fundamental properties.

Lemma 4.d.13. *Let $n, m, k \in \mathbb{N}$.* [289]

1. *For every n we have $0 \le n$*
2. *$n \le m$ if and only if $n < S(m)$.*
   Note that these two points satisfy (4.d.3),(4.d.2) in 4.d.1
3. *For every n we have $n < S(n)$*
4. *$n < m$ if and only if $S(n) \le m$.*

5. *If $n \le m \le S(n)$ then $m = n$ or $m = S(n)$.*

The proofs are left as exercise 4.d.18. (After we will prove that the relation is total, then by 4.d.9 the last two are equivalent.)

Proposition 4.d.14. *$\le$ is an order relation.* [28B]

Proof. Reflexive property: $n + 0 = n$. Antisymmetric property: if $n + k = m$ and $m + h = n$ then $n + k + h = n$ therefore by cancellazione 4.c.9 $h + k = 0$, and for 4.c.10 $h = k = 0$ so $n = m$. Transitive property: if $n + k = m$ and $m + h = p$ then $n + k + h = p$. □

Hence this relation "$\le$" defined in 4.d.12 satisfies the principle 4.d.1; we will show that any such ordering is a well order; here we present though a self contained proof for this specific case. [298]

Proposition 4.d.15. *$\le$ is a total order relation.* [28Z]

Proof. Consider the proposition

$$P(n) \doteq \forall m \in \mathbb{N}, n \le m \vee m \le n$$

then $P(0)$ is true. Let's assume $P(n)$; let's fix an m;

- if $m \le n$ then $m \le S(n)$, by the lemma (point 2), so $P(Sn)$ holds;
- if $\neg m \le n$ but $P(n)$ holds, then $n \le m$ must hold, but it cannot be $n = m$, so $n < m$ holds: but then $S(n) \le m$ by the lemma (point 4);

in any case $P(S(n))$ is proven starting from $P(n)$. □

Proposition 4.d.16. *$\le$ is a well ordering.* [297]

Proof. Trace of proof. By Lemma 4.d.13 (point (2)) we know that this relation satisfies the strong induction principle 4.f.2; so we can prove that any non empty subset has a minimal element as in Esercise 4.f.5; but we know that the ordering is total, so the minimal element is the minimum. □

Definition 4.d.17 (Subtraction)**.** *If $m \ge n$, there exists an unique h such that $m = n + h$ (uniqueness follows from 4.c.9); we will indicate this h as $m - n$.* [28C]

Exercises

E4.d.18 Show properties in 4.d.13. *Hidden solution:* *[UNACCESSIBLE UUID '28F']* [28D]

E4.d.19 Show that if $n \le m$ then $m - n \le m$. *Hidden solution:* *[UNACCESSIBLE UUID '28H']* [28G]

E4.d.20 Show that if $n \ne 0$ then $n \times m \ge m$. *Hidden solution:* *[UNACCESSIBLE UUID '28P']* [28N]

E4.d.21 `Topics:Euclidean division.` [28J]

Prove that, given $d, n \in \mathbb{N}, d \ge 1$, two numbers $q, r \in \mathbb{N}, 0 \le r < d$ exist and are unique for which $n = q \times d + r$ (where n is the "dividend" d is the "divisor", q is the "quotient" and r is the "remainder") *Hidden solution:* *[UNACCESSIBLE UUID '28K']*

E4.d.22 *(Replaces 282)* [28M]

Let $h \neq 0$, prove that if $n \times h = m \times h$ then $n = m$. *(Sugg. use subtraction)*

Hidden solution: [UNACCESSIBLE UUID '28S']

In particular the map $n \mapsto n \times h$ is injective.

§4.d.b Ordering and arithmetic [28Q]

The ordering is compatible with arithmetic.

Proposition 4.d.23. [28R]

- (Addition and ordering compatibility) *You have $n \leq m$ if and only if $n + k \leq m + k$.*
- (Multiplication and ordering compatibility) *When $k \neq 0$ you have $n \leq m$ if and only if $n \times k \leq m \times k$.*

In particular (remembering 4.d.22) the map $n \mapsto n \times h$ is strictly increasing (and hence injective).

Proof. We will use some properties left for exercise.

- If $n \leq m$, by definition $m = n+h$, then $n+k \leq m+k$ because $m+k = n+h+k$ (note that we are using associativity). If $n+k \leq m+k$ let then j the only natural number such that $n + k + j = m + k$ but then $n + j = m$ by cancellation 4.c.9.
- If $n \leq m$ then $m = n + h$ therefore $m \times k = n \times k + h \times k$ so $n \times k \leq m \times k$. Vice versa let $k \neq 0$ and $n \times k \leq m \times k$ i.e. $n \times k + j = m \times k$: divide j by k using the division 4.d.21, we write $j = q \times k + r$ therefore for associativity $(n + q) \times k + r = m \times k$ but for the uniqueness of the division $r = 0$; eventually collecting $(n + q) \times k = m \times k$ and using 4.d.22 we conclude that $(n + q) = m$.

□

§4.e Z-F and Peano compatibility [26F]

Let's go back now to the model $\mathbb{N}_{ZF}$ of $\mathbb{N}$ built relying on the theory of Zermelo—Fraenkel, seen in Sec. §3.h. We want to see that this model satisfies Peano's axioms.

Recall that, given x (any set, not necessarily natural number) the successor is defined as

$$S(x) \stackrel{\text{def}}{=} x \cup \{x\} \quad .$$

It's easy to see that **N1** and **N3** are true. The **N5** property follows from the fact that $\mathbb{N}_{ZF}$ is the smallest set that is S-saturated. **N2** and **N4**, derive from 3.h.8.

We moreover saw in Theorem 3.h.16 that the relation $\subseteq$ satisfies the requisites of Hypothesis 4.d.1.

§4.f Generalized induction, well ordering [27M]

Proposition 4.f.1 (Generalized induction)**.** *Let $N \in \mathbb{N}$, and let $P(n)$ a be logical clause, true for $n = N$ and such that* [1XR]

$$\forall n \geq N, P(n) \Rightarrow P(S(n)) \quad ,$$

then P is true for every $n \geq N$.

Let us now present the principle of strong induction.

Proposition 4.f.2 (Strong Induction)**.** *Assume that a (partial) order associated to $\mathbb{N}$ satisfies 4.d.1. Let $P(n)$ be a logical clause, true for $n = 0$ and such that* [1XS]

$$\forall n \in \mathbb{N}, \Big(\big(\forall k \leq n, P(k) \big) \Rightarrow P(Sn) \Big) \tag{4.f.3}$$

then P is true for every $n \in \mathbb{N}$.

This principle is apparently stronger than the usual one; but we'll see that it is in fact equivalent.

Even this result can be generalized by requiring that $P(N)$ is true, and writing the inductive hypothesis in the form "$\forall k, N \leq k \leq n, P(k)$": you will get that $P(n)$ is true for $n \geq N$.

Note that the principle of well ordering is in some sense equivalent to the principle of induction; see 4.f.8.

Exercises

E4.f.4 `Prerequisites:4.d.1,4.a.1.Difficulty:*.` [1XN]

Use the induction principle 4.a.1 to demonstrate the strong induction principle 4.f.2

Warning: use the properties in Hypothesis 4.d.1, but do not assume that $\leq$ is a total order: indeed this result is needed to prove it.

Hidden solution: *[UNACCESSIBLE UUID '1XQ']*

E4.f.5 `Prerequisites:4.d.1,4.f.2.Difficulty:*.` [1XP]

Assume that a (partial) order $\leq$ associated to $\mathbb{N}$ satisfies 4.d.1. Use the strong induction principle 4.f.2 to show that every non-empty $A \subseteq \mathbb{N}$ contains a minimal element, *i.e.*

$$\exists a \in A \,,\ \forall b \in A \,,\ \neg(b < a) \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '1XZ']*

E4.f.6 `Prerequisites:4.d.6,4.f.5,3.i.6.` Use the prerequisites to prove that $(\mathbb{N}, \leq)$ is well ordered. [273]

E4.f.7 `Prerequisites:4.f.2.` Use strong induction to show that every $n \geq 2$ factorizes into the product of prime numbers. [1XT]

Hidden solution: *[UNACCESSIBLE UUID '1XV']*

E4.f.8 Difficulty:*. Let A be a well-ordered set [†38] by the order $\leq$; let $m = \min A$; then for propositions $P(a)$ with $a \in A$ you can use a proof method, called *transfinite induction*, in which [1XY]

- $P(m)$ is required to be true, and
- the following "inductive step" is proven:

$$\forall n \in \mathbb{N}\Big((\forall k < n, P(k)) \Rightarrow P(n)\Big)$$

Show that if the proposition P satisfies the previous two requirements, then $\forall x \in A, P(x)$.

Prove also that if $A = \mathbb{N}$ then the "inductive step" is equivalent to the inductive step of strong induction (defined in 4.f.2).

Other exercises regarding "induction" are: 9.a.9

§4.g Frequently, eventually [26G]

Let $\mathbb{N}$ be the natural numbers.

Definition 4.g.1 (frequently, eventually). *Let* P(n) *be a logical clause that depends on a free variable* $n \in \mathbb{N}$. *We will say that* [018] (Solved on 2022-10-27)

P(n) *holds eventually in n if*	$\exists m \in \mathbb{N}, \forall n \in \mathbb{N}$ *with* $n \geq m$, P(n) *holds ;*
P(n) *frequently holds in n if*	$\forall m \in \mathbb{N}, \exists n \in \mathbb{N}$ *with* $n \geq m$ *for which* P(n) *holds.*

This definition is equivalent to definition 6.b.2 for real variable $x \to \infty$; it can be further generalized, as seen in 3.d.28.

Remark 4.g.2. *In Italian* frequentemente *(for* frequently*) and* definitivamente *(for* eventually*) are commonly used in text books; whereas in English these terms are not widely used.*[†39] [23Q]

Exercises

E4.g.3 Note that «*P(n)* holds eventually in *n*» implies «*P(n)* holds frequently in *n*». [019]

Hidden solution: [UNACCESSIBLE UUID '01B']

E4.g.4 Note that «(non *P(n)*) holds frequently in *n*» if and only if «non (*P(n)* holds eventually in *n*)». [01C]

Hidden solution: [UNACCESSIBLE UUID '01D']

E4.g.5 Note that «*P(n)* holds frequently in *n*» if and only if «*P(n)* holds for infinitely many *n*». [01F]

(This equivalence is not true in a generic ordered set. See instead 3.d.29 for the correct formulation).

E4.g.6 Let now $P(n), Q(n)$ be two propositions. [01G] (Solved on 2022-10-27// in parte)

[†38] As defined in 3.i.1.
[†39] With some notable exceptions, such as [14]

- Say what implications there are between
  - ”($P(n) \wedge Q(n)$) is valid eventually” and
  - ”$P(n)$ is valid eventually and $Q(n)$ is valid eventually”.
- Similarly for propositions
  - ”($P(n) \vee Q(n)$) is valid eventually” and
  - ”$P(n)$ is valid eventually or $Q(n)$ is valid eventually”.

Also formulate similar results for the notion of ”frequently”.

Hidden solution: [UNACCESSIBLE UUID '01H']

E4.g.7 Let again $P(n), Q(n)$ be two propositions. If *”$P(n)$ is valid eventually* and *$Q(n)$ is valid frequently”* then *”($P(n) \wedge Q(n)$) is valid frequently”*. [29G]

§5 Groups, Rings, Fields [1ZD]

We review these definitions.

Definition 5.1. *A **group** is a set G equipped with a binary operation $*$, that associates an element $a * b \in G$ to each pair $a, b \in G$, respecting these properties.* [1ZF] (Solved on 2022-11-15)

1. *Associative property: for any given $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$.*
2. *Existence of the neutral element: an element denoted by e such that $a*e = e*a = a$.*
3. *Existence of the inverse: each element $a \in G$ is associated with an **inverse element** a', such that $a * a' = a' * a = e$. The inverse of the element a is often denoted by a^{-1} (or $-a$ if the group is commutative).* †40

*A group is said to be **commutative** (or abelian) if moreover $a * b = b * a$ for each pair $a, b \in G$.*

Definition 5.2. *A **ring** is a set A with two binary operations* [1ZG] (Solved on 2022-11-15)

- *$+$ (called sum or addition) and*
- *$\cdot$ (called "multiplication", also indicated by the symbol $\times$ or $*$, and often omitted),*

such that

- *A $+$ is a commutative group (usually the neutral element is denoted by 0);*
- *the operation $\cdot$ has neutral element (usually the neutral element is indicated by 1) and is associative;*
- *multiplication distributes on addition, both on the left*

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \forall a, b, c \in A$$

and on the right

$$(b + c) \cdot a = (b \cdot a) + (c \cdot a) \quad \forall a, b, c \in A$$

*A ring is called **commutative** if multiplication is commutative. (In which case the right or left distributions are equivalent.)*

We assume that $0 \neq 1$ (otherwise $\{0\}$ would be a ring).

Examples of commutative rings are: integer numbers $\mathbb{Z}$, polynomials $A[x]$ with coefficients in a commutative ring A.

An example of a non-commutative ring is given by matrixes $\mathbb{R}^{n\times n}$, with the usual operation of multiplication and addition.

Definition 5.3. *A **field** F is a ring in which multiplication is commutative, and every element $x \in F$ with $x \neq 0$ has an inverse x^{-1} for multiplication.* [1ZH] (Solved on 2022-11-15)

(So $F \setminus \{0\}$ is a commutative group for multiplication, see 5.13).

Some field examples are: rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

Remark 5.4. *Typically*[†41] *you use the notations on the left instead of the writings on the right (where x, y, z are in the field and n is positive integer)* [20R] *(Solved on 2022-11-15)*

$x - y$	$x + (-y)$
$\frac{x}{y}$	$x \cdot y^{-1}$
$x + y + z$	$(x + y) + x$
xyz	$(x \cdot y) \cdot z$
nx	$\underbrace{x + \ldots + x}_{n \text{ times}}$
x^n	$\underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$
x^{-n}	$(x^{-1})^n$

Precisely, nx means "add x to itself n times"; the operation $n \mapsto n \cdot x$ can be defined recursively setting $0 \cdot x = 0$ and $(n + 1) \cdot x = n \cdot x + x$. Similarly x^n means "multiply x by itself n times": see the exercise 5.21.

Remark 5.5. *Hurwitz's theorem [39] asserts that if V is a field and is also a real vector space with a scalar product, then $V = \mathbb{R}$ or $V = \mathbb{C}$.* [1ZW]

Definition 5.6. *An **ordered ring** F is a ring with a total order relation $\le$ for which, for every $x, y, z \in F$,* [1ZJ]

- $x \le y \Rightarrow x + z \le y + z$;
- $x, y \ge 0 \Rightarrow x \cdot y \ge 0$.

Due to 5.13, if F is a field, in the second hypothesis we may equivalently write $x, y > 0 \Rightarrow x \cdot y > 0$. (Regarding the second hypothesis, see also 5.14) For further informations see the references in [32]. We will assume that in an ordered ring the multiplication is commutative.

Examples of ordered field are: rational numbers $\mathbb{Q}$ the real numbers $\mathbb{R}$. The complex numbers $\mathbb{C}$ do not allow an ordering satisfying the above properties (see exercise 5.19).

Definition 5.7. *An ordered field F is **archimedean** if $\forall x, y \in F$ with $x > 0, y > 0$ there is a $n \in \mathbb{N}$ for which $nx > y$. (See 5.4 for the definition of nx).* [1ZK]

[†42]

Exercises

E5.8 The neutral element of a group is unique. *Hidden solution: [UNACCESSIBLE UUID '1ZN']* [1ZM]

E5.9 In a group, the inverse of an element is unique. *Hidden solution: [UNACCESSIBLE UUID '1ZQ']* [1ZP]

E5.10 Having fixed an element $g \in G$ in a group, the left and right multiplications [29C]

[†40]The notation a^{-1} is justified by the fact that the inverse element is unique: *cf* 5.9.
[†41]Taken from 1.13 in [22]
[†42]Parts of the following exercises are from Chap. 2 Sec. 2 in [2], or Chap. 1 in [22].

$L_g : G \to G$ and $R_g : G \to G$

$$L_g(h) = g * h \ , \ R_g(h) = h * g$$

are bijections.

E5.11 Prove[†43] that in a group: [1ZR]

1. If $x + y = x + z$ then $y = z$.
2. If $x + y = x$ then $y = 0$.
3. If $x + y = 0$ then $y = -x$.
4. $-(-x) = x$.

E5.12 Prove[†44] that in a ring: [1ZS]

1. $0 \cdot x = 0$
2. $(-x)y = -(xy) = x(-y)$.
3. $(-x)(-y) = xy$.
4. $(-1)x = -x$.

Hidden solution: *[UNACCESSIBLE UUID '299']*

E5.13 Consider the property [203]

$$\forall x, y \in A \ , \ x \cdot y = 0 \Rightarrow x = 0 \vee y = 0$$

this property may be false in a ring A; if it holds in a specific ring, then this ring is said to be an *integral domain* [41].

Show that a field F is always an integral domain. Consequently $F \setminus \{0\}$ is a commutative group for multiplication. *Hidden solution:* *[UNACCESSIBLE UUID '204']*

E5.14 Suppose that in a ring A there is a total ordering $\le$ such that for every $x, y, z \in A$ you have $x \le y \Rightarrow x + z \le y + z$; then show that these are equivalent [1ZT]

- $x \le y \wedge 0 \le z \quad \Rightarrow \quad x \cdot z \le y \cdot z$;
- $x \ge 0 \wedge y \ge 0 \quad \Rightarrow \quad x \cdot y \ge 0$.

E5.15 Prerequisites:3.d.4,5.12,5.14. Prove [†45] than in an ordered ring F: [1ZV]

1. for each $x \in F, x^2 \ge 0$, in particular $1 = 1^2 > 0$;
2. $x > 0 \Rightarrow -x < 0$
3. $y > x \Rightarrow -y < -x$;
4. $x \le y \wedge a \le 0 \Rightarrow a \cdot x \ge a \cdot y$;
5. $x \ge a \wedge y \ge b \Rightarrow x + y \ge a + b$;
6. $x > a \wedge y \ge b \Rightarrow x + y > a + b$;
7. $x \ge a \ge 0 \wedge y \ge b \ge 0 \Rightarrow x \cdot y \ge a \cdot b$;

[†43] [22] Prop. 1.14
[†44] [22] Prop. 1.16
[†45] From Cap. 2 Sec. 7 in [2], or [22] Prop. 1.18

Prove than in an ordered field F:

1. $x > a > 0 \wedge y > b \geq 0 \Rightarrow x \cdot y > a \cdot b$;
2. $x > 0 \Rightarrow x^{-1} > 0$;
3. $y > x > 0 \Rightarrow x^{-1} > y^{-1} > 0$;
4. $x \cdot y > 0$ if and only if x and y agree on sign (i.e. either both > 0 or both < 0);

Hidden solution: [UNACCESSIBLE UUID '29B']

E5.16 In an ordered field F we call $P = \{x \in F : x \geq 0\}$ the set of positive (or zero) numbers; it satisfies the following properties: †46 [1ZX]

- $x, y \in P \Rightarrow x + y, x \cdot y \in P$,
- $P \cap (-P) = \{0\}$ and
- $P \cup (-P) = F$.

vice versa if in a field F we can find a set $P \subseteq F$ that satisfies them, then F is an ordered field by defining $x \leq y \Leftrightarrow y - x \in P$.

E5.17 Not all fields are infinite sets. Consider $X = \{0, 1\}$ and operations $0 + 0 = 1 + 1 = 0$, $0 + 1 = 1 + 0 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Check that it is a field. Show that it cannot be an ordered field. [1ZY]

E5.18 Consider the ring of matrixes $\mathbb{R}^{2\times 2}$ let's define [1ZZ]

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ,$$

then check that

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ;$$

you conclude that the ring of matrixes is not commutative.

E5.19 Show that there is no ordering $\leq$ on $\mathbb{C}$ such that $(\mathbb{C}, \leq)$ is an ordered field. *Hidden solution:* [UNACCESSIBLE UUID '20S'] [08V]

E5.20 Let's fix an integer $N \geq 2$ that it is not a perfect square. Consider the subset F of $\mathbb{R}$ given by the numbers x that can be written as $x = a + b\sqrt{N}$, with $a, b \in \mathbb{Q}$; we associate the operations of $\mathbb{R}$: show that F is a field. *Hidden solution:* [UNACCESSIBLE UUID '201'] [200]

E5.21 Let F be a field; given $\alpha \neq 0$ and $h \in \mathbb{N}$ consider the recursive definition of exponentiation α^h defined from $\alpha^0 = 1$ and $\alpha^{(n+1)} = \alpha^n \cdot \alpha$; then prove that $\alpha^{h+k} = \alpha^h \alpha^k$ and $(\alpha^h)^k = \alpha^{(hk)}$ for every $k, h \in \mathbb{N}$. [202]

E5.22 Prerequisites:5.21.Given $\alpha \neq 0$ in a field, define that $\alpha^0 = 1$ and let α^{-n} be the multiplicative inverse of α^n when $n \geq 1$ natural. (Use 5.21). For $n, m \in \mathbb{Z}$ show that [20T]

$$\alpha^n \alpha^m = \alpha^{n+m} \quad , \quad (\alpha^h)^k = \alpha^{(hk)} \quad ;$$

if the field is ordered and $\alpha > 1$ show that $n \mapsto \alpha^n$ is strictly monotonic increasing.

E5.23 Let F be a commutative ring, $a, b \in F$, $n \in \mathbb{N}$ then [205]

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where the factor

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!}$$

is called the *"binomial coefficient"*. (This result is known as the binomial theorem, Newton's formula, Newton's binomial). To prove it by induction, check that

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

for $0 \le k, k+1 \le n$.

†46 From Chap. 2 Sect. 7 in [2]

§6 Real line [09X]

We will indicate in the following with $\mathbb{R}$ the real line, and with $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ its extension. †47

We will use *intervals* (see definition in 3.d.44).

Remark 6.1. *Given a set $I \subset \mathbb{R}$ there are various ways of saying that a function $f : I \to \mathbb{R}$ is **monotonic**. Let's first list the different types of monotonicity* [2DJ]

$$\forall x, y \in I, x < y \Longrightarrow f(x) \le f(y) \tag{6.2}$$
$$\forall x, y \in I, x < y \Longrightarrow f(x) < f(y) \tag{6.3}$$
$$\forall x, y \in I, x < y \Longrightarrow f(x) \ge f(y) \tag{6.4}$$
$$\forall x, y \in I, x < y \Longrightarrow f(x) > f(y) \tag{6.5}$$

Unfortunately in common use there are different and incompatible conventions used when naming the previous definitions. Here is a table, in which every convention is a column.

(6.2)	*non-decreasing*	*increasing*	*weakly increasing*
(6.3)	*increasing*	*strictly increasing*	*strictly increasing*
(6.4)	*non-increasing*	*decreasing*	*weakly decreasing*
(6.5)	*decreasing*	*strictly decreasing*	*strictly decreasing*

In this text, the convention in the last column is used.

(The first column is, in my opinion, problematic. It often leads to the use, unfortunately common, of phrases such as *"f is a non-decreasing function"* or *"we take a function f not decreasing"*; this can give rise to confusion: seems to say that f does not meet the requirement to be "decreasing", but it does not specify whether it is monotonic. People who follow the convention in the first column (in my opinion) should always say "monotonic").

Exercises

E6.6 `Prerequisites:3.d.49.` [09Y]

Show that any interval I in $\mathbb{R}$ falls in one of the categories seen in 3.d.45. *Hidden solution: [UNACCESSIBLE UUID '09Z']*

E6.7 `Prerequisites:5.22.`Let $\alpha > 0, \alpha \in \mathbb{R}$ be fixed. We know that, for every natural $n \ge 1$, there exists an unique $\beta > 0$ such that $\beta^n = \alpha$, and β is denoted by the notation $\sqrt[n]{\alpha}$. (See e.g. Proposition 2.6.6 Chap. 2 Sec. 6 of the course notes [2] or Theorem 1.21 in [22]). Given $q \in \mathbb{Q}$, we write $q = n/m$ with $n, m \in \mathbb{Z}, m \ge 1$, we define [20V]

$$\alpha^q \stackrel{\text{def}}{=} \sqrt[m]{\alpha^n} \quad .$$

Show that this definition does not depend on the choice of representation $q = n/m$; that

$$\alpha^q = \left(\sqrt[m]{\alpha}\right)^n \quad ;$$

that for $p, q \in \mathbb{Q}$

$$\alpha^q \alpha^p = \alpha^{p+q} \quad , \quad (\alpha^p)^q = \alpha^{(pq)} \quad ;$$

show that when $\alpha > 1$ then $p \mapsto \alpha^p$ is strictly monotonic increasing.

E6.8 Prerequisites:6.7.Difficulty:*.Having fixed $\alpha > 1$, we define, for $x \in \mathbb{R}$, [20W]

$$\alpha^x = \sup\{\alpha^p : p \in \mathbb{Q}, p \le x\} \quad ;$$

show that:

- this is a good definition (i.e. that the set on the right is bounded above and not empty).
- Iff x is rational then α^x (as above defined) coincides with the definition in the previous exercise 6.7.
- show that $x \mapsto \alpha^x$ is strictly increasing.
- Show that
$$\alpha^x \alpha^y = \alpha^{x+y} \quad , \quad (\alpha^x)^y = \alpha^{(xy)} \quad .$$

See also the exercise 14.a.5.

E6.9 Let $a, b \in \mathbb{R}$ be such that [20X]

$$\forall L \in \mathbb{R}, L > b \Rightarrow L > a \quad .$$

Prove that $b \ge a$.

E6.10 Fix $I = \{1, \dots n\}$. Let n distinct points $y_1, \dots y_n \in \mathbb{R}$ be given; let $\sigma : I \to I$ be a permutation for which triangle inequalities between successive points are equalities *i.e.* [0B0]

$$|y_{\sigma(i+2)} - y_{\sigma(i+1)}| + |y_{\sigma(i+1)} - y_{\sigma(i)}| = |y_{\sigma(i+2)} - y_{\sigma(i)}|$$

for $i = 1, \dots n-2$. Show that there are only two, we call them σ_1, σ_2. Tip: Show that any such permutation necessarily puts the points "in order", *i.e.* you have

$$\forall i, y_{\sigma_1(i+1)} > y_{\sigma_1(i)} \quad , \quad \forall i, y_{\sigma_2(i+1)} < y_{\sigma_2(i)}$$

(up to deciding which is σ_1 and which is σ_2).

Hidden solution: *[UNACCESSIBLE UUID '0B1']*

§6.a Neighbourhoods

[29H] *(Solved on 2022-11-24)*

Neighbourhoods are a family of sets associated with a point $x_0 \in \mathbb{R}$, or $x_0 = \pm\infty$. The neighbourhoods are sets that contain an "example" set. Let's see here some definitions.

Definition 6.a.1 (Neighbourhoods). *The* deleted neighbourhoods *(sometimes called* punctured neighbourhoods*) of points $x_0 \in \mathbb{R}$ are divided into three classes.* [0B2]

- *Neighborhoods of $x_0 \in \mathbb{R}$, which contain a set of the type $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ for a $\delta > 0$;*
- *right neighborhoods of $x_0 \in \mathbb{R}$, which contain a set of the type $(x_0, x_0 + \delta)$ for a $\delta > 0$;*
- *left neighborhoods of $x_0 \in \mathbb{R}$, which contain a set of the type $(x_0 - \delta, x_0)$ for a $\delta > 0$;*

†47 The topological structure of $\overline{\mathbb{R}}$ will be discussed further in 8.b.2.

In any case, the deleted neighborhoods must not contain the point x_0. The "full" neighborhoods are obtained by adding x_0. The "full neighborhoods" are the base for the standard topology on $\mathbb{R}$.

To the previous ones we then add the neighborhoods of $\pm\infty$:

- *neighborhoods of ∞ , which contain a set of the type (y, ∞) as $y \in \mathbb{R}$ varies;*
- *neighborhoods of $-\infty$, which contain a set of the type $(-\infty, y)$ as $y \in \mathbb{R}$ varies;*

In this case we do not distinguish "deleted" neighborhoods and "full" neighborhoods.

Exercise 6.a.2. *Prerequisites:3.d.13.Difficulty:*.* *Let $x_0 \in \overline{\mathbb{R}}$ and $\mathcal{F}$ all the neighbourhoods of x_0. We associate the ordering* [29J] (Proposed on 2022-11-24)

$$I, J \in \mathcal{F} \ \ , I \le J \iff I \supseteq J$$

show that this is a filtering ordering.

(This holds both for "deleted" and for "full" neighbourhoods; for 'left", "right", or "bilateral" neighbourhoods).

(See also 8.15 for the similar statement in topological spaces).

§6.b Frequently, eventually [29K]

We will write $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{\pm\infty\}$.

Definition 6.b.1 (accumulation point)**.** *Given $A \subseteq \mathbb{R}$, a point $x \in \overline{\mathbb{R}}$ is called* accumulation point *for A if every deleted neighborhood of x intersects A.* [0BG]

Definition 6.b.2 (frequently, eventually)**.** *Let $I \subseteq \mathbb{R}$ be a set, $x_0 \in \overline{\mathbb{R}}$ an accumulation point for I. Let* P(x) *be a logical proposition that we can evaluate for $x \in I$. We define that* [0B3]

"P(x) *holds eventually for x tending to x_0" if*	*there is a neighborhood U of x_0 $\forall x \in U \cap I$,* P(x) *is true ;*
"P(x) *frequently holds for x tending to x_0" if*	*for every neighborhood U of x_0 $\exists x \in U \cap I$ for which* P(x) *;*

where it is meant that the neighbourhoods are "deleted".

Remark 6.b.3. *As already seen in 4.g.4, again in this case the following two propositions are equivalent.* [0B4]

- *"not $\Big($ $P(x)$ definitely applies, for x tending to x_0 $\Big)$",*
- *" $\big($ not $P(x)$ $\big)$ frequently applies, for x tending to x_0".*

Remark 6.b.4. *If $x_0 \in \overline{\mathbb{R}}$ is not an accumulation point for I, then we always have that "$P(x)$* definitely is true, for x tending to x_0" . [22X]

Proposition 6.b.5. *Suppose for simplicity that $I = \mathbb{R}$. Putting together the previous ideas, we can write equivalently:* [20C]

- *if $x_0 \in \mathbb{R}$,*

$\exists\delta > 0, \forall x \neq x_0, \lvert x - x_0\rvert < \delta \Rightarrow P(x)$	*$P(x)$ definitely applies for x tending to x_0*
$\forall\delta > 0, \exists x \neq x_0, \lvert x - x_0\rvert < \delta \wedge P(x)$	*$P(x)$ frequently applies for x tending to x_0*

- *whereas in case $x_0 = \infty$*

$\exists y \in \mathbb{R}, \forall x, x > y \Rightarrow P(x)$	*$P(x)$ definitely applies for x tending to ∞*
$\forall y \in \mathbb{R}, \exists x, x > y \wedge P(x)$	*$P(x)$ frequently applies for x tending to ∞*

- *and similarly $x_0 = -\infty$*

$\exists y \in \mathbb{R}, \forall x, x < y \Rightarrow P(x)$	*$P(x)$ definitely applies for x tending to $-\infty$*
$\forall y \in \mathbb{R}, \exists x, x < y \wedge P(x)$	*$P(x)$ frequently applies for x tending to $-\infty$*

§6.c Supremum and infimum [29M]

Let's first review the characterizations of the supremum and infimum in $\mathbb{R}$, as seen in Sec. §3.d.c (or in Chap. 1 Sect. 5 in the notes [2]). Let $A \subseteq \mathbb{R}$ be a non empty set.

Definition 6.c.1. *Let $A \subseteq \mathbb{R}$ be not be empty. Recall that the **supremum**, or **least upper bound**, of a set A is the minimum of majorants; We will indicate it with the usual writing $\sup A$. If A is bounded above then $\sup A$ is a real number; otherwise, by convention, it is set to $\sup A = +\infty$.* [08T]

Proposition 6.c.2. *Let therefore $A \subseteq \mathbb{R}$ be not empty, let $l \in \mathbb{R} \cup \{+\infty\}$; you can easily demonstrate the following properties:* [208] *(Solved on 2022-11-24)*

$\sup A \le l$	$\forall x \in A, x \le l$
$\sup A > l$	$\exists x \in A, x > l$
$\sup A < l$	$\exists h < l, \forall x \in A, x \le h$
$\sup A \ge l$	$\forall h < l, \exists x \in A, x > h$

the first and third derive from the definition of supremum, †48 *the second and fourth by negation; in the third we can conclude equivalently that $x < h$, and in the fourth that $x \ge h$.*

If $l \ne +\infty$ then also we can also write (replacing $h = l - \varepsilon$)

$\sup A < l$	$\exists \varepsilon > 0, \forall x \in A, x \le l - \varepsilon$
$\sup A \ge l$	$\forall \varepsilon > 0, \exists x \in A, x > l - \varepsilon$

Combining the previous results, we get the result already seen in 3.d.41

Corollary 6.c.3. *Having fixed a set $A \subseteq \mathbb{R}$ not empty, then $\sup A$ is the only number $\alpha \in \mathbb{R} \cup \{+\infty\}$ which satisfies these two properties* [20K] *(Solved on 2022-11-24)*

$$\forall x \in A, x \le \alpha$$
$$\forall h < \alpha, \exists x \in A, x > h$$

as already seen in 3.d.41 for the more general case of totally ordered sets.

Definition 6.c.4. *Similarly, given $A \subseteq \mathbb{R}$ not empty,* the greatest lower boundary, *or* infimum, *of A is the maximum of minorants; we will indicate it with the usual writing* $\inf A$. *If A is bounded below then* $\inf A$ *is a real number; otherwise, by convention, we set* $\inf A = -\infty$. [209]

Remark 6.c.5. *Note that if we replace A with* [0B5] *(Proposed on 2022-11-24)*

$$-A = \{-x : x \in A\}$$

and l with $-l$, we switch from the definitions of sup *to those of* inf *(and vice versa).*

Proposition 6.c.6. *Let $A \subseteq \mathbb{R}$ not empty, let $l \in \mathbb{R} \cup \{-\infty\}$; the following properties apply:* [20B]

$\inf A \geq l$	$\forall x \in A, x \geq l$
$\inf A < l$	$\exists x \in A, x < l$
$\inf A > l$	$\exists h > l, \forall x \in A, x \geq h$
$\inf A \leq l$	$\forall h > l, \exists x \in A, x < h$
If $l \neq -\infty$ then also we write (substituting $h = l + \varepsilon$)	
$\inf A > l$	$\exists \varepsilon > 0, \forall x \in A, x \geq l + \varepsilon$
$\inf A \leq l$	$\forall \varepsilon > 0, \exists x \in A, x \leq l + \varepsilon$

Corollary 6.c.7. *Having fixed $A \subseteq \mathbb{R}$ not empty, then $\inf A$ is the only number $\alpha \in \mathbb{R} \cup \{-\infty\}$ which satisfies these two properties* [20M] *(Proposed on 2022-11-24)*

$$\forall x \in A, x \geq \alpha$$
$$\forall h > \alpha, \exists x \in A, x < h$$

Often the above definitions and properties are used in this form.

Definition 6.c.8. *Given J an index set (not empty), let $a_n \in \mathbb{R}$ for $n \in J$. The supremum and infimum are defined as* [20H] *(Solved on 2022-11-24)*

$$\sup_{n\in J} a_n = \sup A \quad , \quad \inf_{n\in J} a_n = \inf A$$

where $A = \{a_n : n \in J\}$ is the image of the sequence.

Given D not empty, let $f : D \to \mathbb{R}$ be a function. The supremum and infimum are defined as

$$\sup_{x\in D} f(x) = \sup A \quad , \quad \inf_{x\in D} f(x) = \inf A$$

where $A = \{f(x) : x \in D\}$ is the image of the function.

§6.c.a Exercises

Let I, J be generic non-empty sets. See definitions in Sec. §6.c

Exercises

E6.c.9 Let a_n be a real-valued sequence, for $n \in I$ a set of indexes; let $r > 0, t \in \mathbb{R}, \rho < 0$; show that [0B6] *(Solved on 2022-11-24)*

$$\sup_{n\in I}(a_n + t) = t + \sup_{n\in I} a_n \quad , \quad \sup_{n\in I}(r a_n) = r \sup_{n\in I} a_n \quad , \quad \sup_{n\in I}(\rho a_n) = \rho \inf_{n\in I} a_n \quad .$$

Hidden solution: [UNACCESSIBLE UUID '22W']

E6.c.10 Let $a_{n,m}$ be a real sequence with two indices $n \in I, m \in J$, show that [0B7] *(Solved on 2022-11-24)*

$$\sup_{n\in I, m\in J} a_{n,m} = \sup_{n\in I}\Big(\sup_{m\in J} a_{n,m}\Big) \quad .$$

Hidden solution: [UNACCESSIBLE UUID '0B8']

E6.c.11 Prerequisites:6.c.10,6.c.9. Let a_n, b_n be real sequences, for $n \in I$, show that [0B9] *(Solved on 2022-11-24)*

$$\sup_{n,m\in I}(a_n + b_m) = (\sup_{n\in I} a_n) + (\sup_{n\in I} b_n) \ ,$$

but

$$\sup_{n\in I}(a_n + b_n) \le (\sup_{n\in I} a_n) + (\sup_{n\in I} b_n) \ ;$$

find a case where inequality is strict. *Hidden solution:* *[UNACCESSIBLE UUID '0BB']*

E6.c.12 Prerequisites:6.c.10. Let $A, B \subseteq \mathbb{R}$ and let [0BC]

$$A \oplus B = \{x + y : x \in A, y \in B\}$$

the *Minkowski sum* [†49] of the two sets: show that

$$\sup(A \oplus B) = (\sup A) + (\sup B) \ .$$

Hidden solution: *[UNACCESSIBLE UUID '0BD']*

E6.c.13 Let $I_n \subseteq \mathbb{R}$ (for $n \in \mathbb{N}$) be closed and bounded non-empty intervals, such that $I_{n+1} \subseteq I_n$: show that $\bigcap_{n=0}^{\infty} I_n$ is not empty. [0BF]

This result is known as ***Cantor's intersection theorem*** *[36]. It is valid in more general contexts, see 10.j.11 and 8.d.4.*

If we replace $\mathbb{R}$ with $\mathbb{Q}$ and assume that $I_n \subseteq \mathbb{Q}$, is the result still valid?

E6.c.14 Study the equivalences in proposition 6.c.2 for the case in which $\sup A = +\infty$: What do the formulas on the right say? [20P] *(Solved on 2022-11-24)*

E6.c.15 Rewrite the properties of the clause 6.c.6 for the cases seen in 6.c.8. [20J]

E6.c.16 Calculate supremum and infimum of the following sets (where n, m are integers). [20Y] *(Proposed on 2022-12)*

$$\left\{\frac{mn}{m^2+n^2} : n, m \ge 1\right\} \quad , \quad \left\{\frac{mn}{m+n} : n, m \ge 1\right\}$$
$$\{2^n + 2^m : n, m \in \mathbb{N}\} \quad , \quad \{2^n + 2^m : n, m \in \mathbb{Z}\}$$
$$\left\{\frac{m^2-2}{n} : n, m \in \mathbb{Z}, n \ne 0\right\} \quad , \quad \left\{\frac{m+1}{m^2} : m \in \mathbb{Z}, m \ne 0\right\}$$

§6.d Limits [29N]

We will write $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{\pm\infty\}$.

Definition 6.d.1. *Let $I \subset \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ accumulation point of I, $f : I \to \mathbb{R}$ function, $l \in \overline{\mathbb{R}}$.* [20D]

The idea of limit (right or left or bilateral) is thus expressed.

$\lim_{x\to x_0} f(x) = l$	*for every "full" neighbourhood V of l, there exists a "deleted" neighbourhood U of x_0 such that for every $x \in U \cap I$, you have $f(x) \in V$*

where the neighborhood U will be "right" or "left' if the limit is "right" or "left"; it

[†48] In particular in the third you can think that $h = \sup A$.
[†49] The *Minkowski sum* will return in the section §12.f.

can also be said that

$\lim_{x\to x_0} f(x) = l$	*for every "full" neighbourhood V of l, you have $f(x) \in V$ eventually for x tending to x_0*

adding that $x > x_0$ if the limit is "right", or $x < x_0$ if the limit is "left".

Let us now write these ideas explicitly.

Proposition 6.d.2. *Let I be a set, $x_0 \in \mathbb{R}$ accumulation point for I, $f : I \to \mathbb{R}$ function, $l \in \mathbb{R}$.* [0BH]

Putting together all the definitions seen above, we get these definitions of limit.

In the case $x_0 \in \mathbb{R}$ and $l \in \mathbb{R}$:

$\lim_{x\to x_0} f(x) = l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I \Rightarrow \lvert f(x) - l\rvert < \varepsilon$
$\lim_{x\to x_0^+} f(x) = l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I \Rightarrow \lvert f(x) - l\rvert < \varepsilon$
$\lim_{x\to x_0^-} f(x) = l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I \Rightarrow \lvert f(x) - l\rvert < \varepsilon$

Be $x_0 \in \mathbb{R}$, $l = \pm\infty$.

$\lim_{x\to x_0} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I \Rightarrow f(x) > z$
$\lim_{x\to x_0} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I \Rightarrow f(x) < z$
$\lim_{x\to x_0^+} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) > z$
$\lim_{x\to x_0^+} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) < z$
$\lim_{x\to x_0^-} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) > z$
$\lim_{x\to x_0^-} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) < z$

Let $l \in \mathbb{R}$, $x_0 = \pm\infty$.

$\lim_{x\to\infty} f(x) = l$	$\forall\varepsilon > 0, \exists y, \forall x, x > y, x \in I \Rightarrow \lvert f(x) - l\rvert < \varepsilon$
$\lim_{x\to-\infty} f(x) = l$	$\forall\varepsilon > 0, \exists y, \forall x, x < y, x \in I \Rightarrow \lvert f(x) - l\rvert < \varepsilon$
$\lim_{x\to\infty} f(x) = \infty$	$\forall z, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) > z$
$\lim_{x\to-\infty} f(x) = \infty$	$\forall z, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) > z$
$\lim_{x\to\infty} f(x) = -\infty$	$\forall z, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) < z$
$\lim_{x\to-\infty} f(x) = -\infty$	$\forall z, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) < z$

Remark 6.d.3. *Note that if you replace $f \mapsto -f$, you switch from definitions with $l = \infty$ to those of $l = -\infty$ (and vice versa). Another symmetry is achieved by switching $x_0 \to -x_0$ and the right and left neighbourhoods.* [0BJ]

§6.e Upper and lower limits [29P]

From the previous definition we move on to the definitions of *"limit superior"* lim sup and *"limit inferior"* lim inf. The idea is so expressed.

Definition 6.e.1. *Let $I \subset \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ accumulation point of I, $f : I \to \mathbb{R}$ function. We define* [20F]

$$\limsup_{x\to x_0} f(x) = \inf_{U \text{neighbourhood of} x_0} \sup_{x\in U\cap I} f(x) \tag{6.e.2}$$

$$\liminf_{x\to x_0} f(x) = \sup_{U \text{neighbourhood of} x_0} \inf_{x\in U\cap I} f(x) \tag{6.e.3}$$

where the first "inf" (resp. the "sup") is performed with respect to the family of all the deleted neighbourhoods U of x_0; and the neighbourhoods will be right or left neighbourhoods if the limit is from right or left.

Remark 6.e.4. *Using the properties of* inf, sup, *we obtain for example these characterizations* [20G] *(Solved on 2022-11-29)*

$$\limsup_{x\to x_0} f(x) \le l \iff \forall z > l, \text{ eventually for } x \to x_0, f(x) < z \ ;$$

$$\limsup_{x\to x_0} f(x) \ge l \iff \forall z < l, \text{frequently for } x \to x_0, f(x) > z \ ;$$

and so on. (In this simplified writing, we assume that $x \in I$).

In particular, defining $l = \limsup_{x\to x_0} f(x)$, the previous formulas characterize exactly the "limsup".

Corollary 6.e.5. *You have* $\alpha = \limsup_{x\to x_0} f(x)$ *if and only if* [20N]

$$\forall z > \alpha, \text{ eventually for } x \to x_0, f(x) < z \ ;$$

$$\forall z < \alpha, \text{ frequently for } x \to x_0, f(x) > z \ .$$

We make them explicit further in what follows. (It is recommended to try to rewrite autonomously some items, by way of exercise).

Proposition 6.e.6. *In the case* $x_0 \in \mathbb{R}$ *and* $l \in \mathbb{R}$, *we divide the definition into two conditions:* [†50] [0BK]

$\limsup_{x\to x_0} f(x) \le l$ $\limsup_{x\to x_0} f(x) \ge l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x \ne x_0, x \in I \Rightarrow f(x) < l+\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x \ne x_0, x \in I, f(x) > l-\varepsilon$
$\limsup_{x\to x_0^+} f(x) \le l$ $\limsup_{x\to x_0^+} f(x) \ge l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) < l+\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x > x_0, x \in I, f(x) > l-\varepsilon$
$\limsup_{x\to x_0^-} f(x) \le l$ $\limsup_{x\to x_0^-} f(x) \ge l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) < l+\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x < x_0, x \in I, f(x) > l-\varepsilon$

$\liminf_{x\to x_0} f(x) \ge l$ $\liminf_{x\to x_0} f(x) \le l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x \ne x_0, x \in I \Rightarrow f(x) > l-\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x \ne x_0, x \in I, f(x) < l+\varepsilon$
$\liminf_{x\to x_0^+} f(x) \ge l$ $\liminf_{x\to x_0^+} f(x) \le l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) > l-\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x > x_0, x \in I, f(x) < l+\varepsilon$
$\liminf_{x\to x_0^-} f(x) \ge l$ $\liminf_{x\to x_0^-} f(x) \le l$	$\forall\varepsilon > 0, \exists\delta > 0, \forall x, \lvert x-x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) > l-\varepsilon$ $\forall\varepsilon > 0, \forall\delta > 0, \exists x, \lvert x-x_0\rvert < \delta, x < x_0, x \in I, f(x) < l+\varepsilon$

[†50] In the following tables all commas "," after the last quantifier should be interpreted as conjunctions "$\land$", but were written as "," for lighten the notation.

In the case $x_0 \in \mathbb{R}$ and $l = \pm\infty$:

$\limsup_{x\to x_0} f(x) = \infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I, f(x) > z$
$\limsup_{x\to x_0^+} f(x) = \infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I, f(x) > z$
$\limsup_{x\to x_0^-} f(x) = \infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I, f(x) > z$
$\limsup_{x\to x_0} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I \Rightarrow f(x) < z$
$\limsup_{x\to x_0^+} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) < z$
$\limsup_{x\to x_0^-} f(x) = -\infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) < z$
$\liminf_{x\to x_0} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I \Rightarrow f(x) > z$
$\liminf_{x\to x_0^+} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I \Rightarrow f(x) > z$
$\liminf_{x\to x_0^-} f(x) = \infty$	$\forall z, \exists\delta > 0, \forall x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I \Rightarrow f(x) > z$
$\liminf_{x\to x_0} f(x) = -\infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x \neq x_0, x \in I, f(x) < z$
$\liminf_{x\to x_0^+} f(x) = -\infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x > x_0, x \in I, f(x) < z$
$\liminf_{x\to x_0^-} f(x) = -\infty$	$\forall z, \forall\delta > 0, \exists x, \lvert x - x_0\rvert < \delta, x < x_0, x \in I, f(x) < z$

In the case $x_0 = \pm\infty$ and $l = \pm\infty$:

$\limsup_{x\to\infty} f(x) = \infty$	$\forall z, \forall y, \exists x, x > y, x \in I, f(x) > z$
$\limsup_{x\to-\infty} f(x) = \infty$	$\forall z, \forall y, \exists x, x < y, x \in I, f(x) > z$
$\limsup_{x\to\infty} f(x) = -\infty$	$\forall z, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) < z$
$\limsup_{x\to-\infty} f(x) = -\infty$	$\forall z, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) < z$
$\liminf_{x\to\infty} f(x) = \infty$	$\forall z, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) > z$
$\liminf_{x\to-\infty} f(x) = \infty$	$\forall z, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) > z$
$\liminf_{x\to\infty} f(x) = -\infty$	$\forall z, \forall y, \exists x, x > y, x \in I, f(x) < z$
$\liminf_{x\to-\infty} f(x) = -\infty$	$\forall z, \forall y, \exists x, x < y, x \in I, f(x) < z$

In the case $x_0 = \pm\infty$ and $l \in \mathbb{R}$:

$\limsup_{x\to\infty} f(x) \le l$ $\limsup_{x\to\infty} f(x) \ge l$	$\forall\varepsilon > 0, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) < l + \varepsilon$ $\forall\varepsilon > 0, \forall y, \exists x, x > y, x \in I, f(x) > l - \varepsilon$
$\limsup_{x\to-\infty} f(x) \le l$ $\limsup_{x\to-\infty} f(x) \ge l$	$\forall\varepsilon > 0, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) < l + \varepsilon$ $\forall\varepsilon > 0, \forall y, \exists x, x < y, x \in I, f(x) > l - \varepsilon$
$\liminf_{x\to\infty} f(x) \le l$ $\liminf_{x\to\infty} f(x) \ge l$	$\forall\varepsilon > 0, \forall y, \exists x, x > y, x \in I, f(x) < l + \varepsilon$ $\forall\varepsilon > 0, \exists y, \forall x, x > y, x \in I \Rightarrow f(x) > l - \varepsilon$
$\liminf_{x\to-\infty} f(x) \le l$ $\liminf_{x\to-\infty} f(x) \ge l$	$\forall\varepsilon > 0, \forall y, \exists x, x < y, x \in I, f(x) < l + \varepsilon$ $\forall\varepsilon > 0, \exists y, \forall x, x < y, x \in I \Rightarrow f(x) > l - \varepsilon$

Remark 6.e.7. *Note that* [OBM]

$$\liminf_{x\to x_0} f(x) = \infty \iff \lim_{x\to x_0} f(x) = \infty$$

and

$$\limsup_{x\to x_0} f(x) = -\infty \iff \lim_{x\to x_0} f(x) = -\infty$$

Remark 6.e.8. *Note that if you replace $f \mapsto -f$, $l \mapsto -l$, you pass from the definitions of* lim sup *to those of* lim inf *(and vice versa). Another symmetry is achieved by switching $x_0 \to -x_0$ and right and left neighbourhoods/limits.* [OBN]

Exercises

E6.e.9 Let $A_1, A_2 \dots$ be sets , for $n \in \mathbb{N}$; let $X = \bigcup_n A_n$. We define the characteristic function $\mathbb{1}_A : X \to \mathbb{R}$ as [0BP]

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

We will use the definitions $\limsup_n A_n$ and $\liminf_n A_n$ seen in eqn. (3.k.9) and (3.k.10). You have

$$\mathbb{1}_{(\limsup_n A_n)} = \limsup_n \mathbb{1}_{A_n} , \quad (6.e.10)$$

$$\mathbb{1}_{(\liminf_n A_n)} = \liminf_n \mathbb{1}_{A_n} . \quad (6.e.11)$$

E6.e.12 We fix a real valued sequence a_n. Now consider the definition of 6.e.1 setting $I = \mathbb{N}$ and $x_0 = \infty$, so that neighborhoods of x_0 are sets containing $[n, \infty) = \{m \in \mathbb{N} : m \ge n\}$; with these assumptions show that you have [0BQ]

$$\limsup_{n\to\infty} a_n = \inf_n \sup_{m\ge n} a_n = \lim_{n\to\infty} \sup_{m\ge n} a_n ,$$

$$\liminf_{n\to\infty} a_n = \sup_n \inf_{m\ge n} a_n = \lim_{n\to\infty} \inf_{m\ge n} a_n ., \quad (6.e.13)$$

E6.e.14 Prerequisites:6.e.1,6.a.2,3.d.13,7.d.4,7.d.9.Difficulty:*. [29R] (Proposed on 2022-11-24)

Let $I \subset \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ accumulation point of I, $f : I \to \mathbb{R}$ function. As in 6.a.2 $\mathcal{F}$ all the neighbourhoods of x_0 with associated the filtering ordering

$$U, V \in \mathcal{F} \ , U \le V \iff U \supseteq V \quad .$$

Let

$$s, i : \mathcal{F} \to \mathbb{R} \ , \ s(U) = \sup_{x\in U\cap I} f(x) \ , \ i(U) = \inf_{x\in U\cap I} f(x)$$

note that these are monotonic functions, and show that [†51]

$$\limsup_{x\to x_0} f(x) \stackrel{\text{def}}{=} \inf_{U\in\mathcal{F}} s(U) = \lim_{U\in\mathcal{F}} s(U) \quad (6.e.15)$$

$$\liminf_{x\to x_0} f(x) \stackrel{\text{def}}{=} \sup_{U\in\mathcal{F}} i(U) = \lim_{U\in\mathcal{F}} i(U) \quad (6.e.16)$$

where the limits are defined in 7.d.4.

E6.e.17 Prerequisites:6.e.14. [29S] (Solved on 2022-11-24)

Let $I \subset \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ accumulation point of I, and $f, g : I \to \mathbb{R}$ functions. Prove that

$$\limsup_{x\to x_0} \big(f(x) + g(x)\big) \le \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x) \quad .$$

E6.e.18 Let $I \subset \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ accumulation point of I, $f : I \to \mathbb{R}$ function. Let $r > 0, t \in \mathbb{R}, \rho < 0$; show that [29T]

$$\limsup_{x\to x_0}(f(x) + t) = t + \limsup_{x\to x_0} f(x) \ , \ \limsup_{x\to x_0}(rf(x)) = r \limsup_{x\to x_0} f(x) \ ,$$

$$\limsup_{x\to x_0}(\rho f(x)) = \rho \liminf_{x\to x_0} f(x) \ .$$

Other exercises on limits of sequences can be found in Sec. §7.a.

[†51] *cf* 6.e.2, (6.e.3).

§6.f Approximation of irrational numbers [29Q]

In the next exercises we will use the following definitions.

Definition 6.f.1. *For $x \in \mathbb{R}$ we define $\lfloor x \rfloor$ to be the* ***floor function*** *defined as the greatest integer less than or equal to x, as in* [0BS]

$$\lfloor x \rfloor \stackrel{def}{=} \max\{n \in \mathbb{Z} : n \le x\} \quad .$$

Definition 6.f.2. *$x - \lfloor x \rfloor$ is the fractional part of x.* [0BT]

(We define $\varphi(x) = x - \lfloor x \rfloor$, note that $\varphi(3,1415) = 0,1415$ but $\varphi(-4,222) = 0,778$ because $\lfloor -4,222 \rfloor = -5$).

Exercises

E6.f.3 Note that $k = \lfloor x \rfloor$ is the only integer for which you have $k \le x < k+1$ or equivalently $0 \le (x-k) < 1$ or equivalently $x - 1 < k \le x$. [0BV]

E6.f.4 Prerequisites:6.f.1.Given $x \in \mathbb{R}$ and $N \in \mathbb{N}, N \ge 2$, prove that at least one element of the set $\{x, 2x, \dots, (N-1)x\}$ is at most distance $1/N$ from an integer, that is, there exist $n, m \in \mathbb{Z}$ with $1 \le n \le N-1$ such that $|nx - m| \le 1/N$. [0BW]

Hidden solution: *[UNACCESSIBLE UUID '0BX']*

E6.f.5 Prerequisites:6.f.1,6.f.4.Given $x, b \in \mathbb{R}$ with $x \ne 0$ irrational, and $\varepsilon > 0$, prove that there is a natural M such that $Mx - b$ is at most ε from an integer. [0BY]

Let $\varphi(x) = x - \lfloor x \rfloor$ be the fractional part of x, we have $\varphi(x) \in [0,1)$. The above result implies that the sequence $\varphi(nx)$ is dense in the interval $[0,1]$.

Note that instead if $x \ne 0$ is rational *i.e.* $x = n/d$ with n, d coprime integers and $d > 0$, then the sequence $\varphi(nx)$ assumes all and only the values $\{0, 1/d, 2/d, \dots (d-1)/d\}$.

(This is demonstrated by the Bézout's lemma [35]).

Hidden solution: *[UNACCESSIBLE UUID '0BZ']*

E6.f.6 Prerequisites:6.f.4. *(Dirichlet's approximation theorem)* Given an irrational number x, show that there are infinitely many rationals α such that we can represent $\alpha = m/n$ in order to satisfy the relation [0C1]

$$\left| x - \frac{m}{n} \right| < \frac{1}{n^2} \quad .$$

Some comments.

- Note for every fixed $n \ge 2$ there is at most an m for which the previous relation holds; but there may not be one.
- Note that if the relation holds for a rational α, there are only finite choices of representations for which it holds,
- and certainly it holds for the "canonical" representation with n, m coprimes.

Hidden solution: [UNACCESSIBLE UUID '0C2']

Note that Hurwitz's theorem [40] states that for every irrational number ξ there are infinitely many coprime integers $m, n \in \mathbb{Z}$ such that [2B0]

$$\left|\xi - \frac{m}{n}\right| < \frac{1}{\sqrt{5}\, n^2}.$$

E6.f.7 Fixed $k > 0$, $\varepsilon > 0$ and a rational number x, prove that there exist only finitely many rationals α that can be represented as $\alpha = m/n$ in order to satisfy the relation [0C3]

$$\left|x - \frac{m}{n}\right| \le \frac{k}{n^{1+\varepsilon}} \quad .$$

Hidden solution: [UNACCESSIBLE UUID '0C4']

E6.f.8 Prove that for every rational m/n you have [0C5]

$$\left|\sqrt{2} - \frac{m}{n}\right| > \frac{1}{4n^2}.$$

We obtain that the set $A = \bigcup_{m\in\mathbb{Z}, n\in\mathbb{N}^*} \left(\frac{m}{n} - \frac{1}{4n^2}, \frac{m}{n} + \frac{1}{4n^2}\right)$ is an open set that contains every rational number, but $A \neq \mathbb{R}$.

Hidden solution: [UNACCESSIBLE UUID '0C6']

§6.g Algebraic

Definition 6.g.1. *A number $\alpha \in \mathbb{R}$ is said* algebraic *if there exists a polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with rational coefficients such that $p(\alpha) = 0$. Otherwise α is said* transcendental. [0C7]

We note that every rational $\alpha = n/m$ is algebraic, as the root of $p(x) = mx - n$.

Definition 6.g.2. *Given a commutative ring A, the set of polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with coefficients $a_i \in A$ is usually denoted by $A[x]$; this set, endowed with the usual operations of sum and product of polynomials, is a commutative ring.* [0C8]

We want to show that algebraic numbers are a field.

Exercises

E6.g.3 Given $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $p \in \mathbb{Q}[z]$ such that $p(\alpha) = 0$, build a polynomial $q \in \mathbb{Z}[z]$ such that $q(\alpha) = 0$. [0C9]

So the definition of *algebraic* can be given equivalently with polynomials with integer coefficients.

E6.g.4 Given $\alpha \neq 0$ and $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $p \in \mathbb{Q}[z]$ such that $p(\alpha) = 0$, build a polynomial $q \in \mathbb{Q}[z]$ such that $q(1/\alpha) = 0$. [0CB]

So if $\alpha \neq 0$ is algebraic then $1/\alpha$ is algebraic.

Hidden solution: [UNACCESSIBLE UUID '0BR']

E6.g.5 Given $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $p \in \mathbb{Q}[z]$ such that $p(\alpha) = 0$, given $b \in \mathbb{Q}$ build a $q \in \mathbb{Q}[z]$ such that $q(b\alpha) = 0$. [0CC]

So if α is algebraic then $b\alpha$ is algebraic.

E6.g.6 Given $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $p \in \mathbb{Q}[z]$ such that $p(\alpha) = 0$, given $b \in \mathbb{Q}$ build a $q \in \mathbb{Q}[z]$ such that $q(b + \alpha) = 0$. [0CD]

So if α is algebraic then $b + \alpha$ is algebraic.

E6.g.7 Difficulty:*.More generally, given $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $p \in \mathbb{Q}[z]$ $q(x) = b_0 + b_1 x + \cdots + b_m x^m$, $q \in \mathbb{Q}[z]$, and given α, β such that $p(\alpha) = 0 = q(\beta)$, construct a polynomial $r \in \mathbb{Q}[z]$ such that $r(\alpha + \beta) = 0$. [0CF]

(Hint: use the theory of the resultant [44]).

So if α, β are algebraic then $\alpha + \beta$ is algebraic.

Hidden solution: *[UNACCESSIBLE UUID '0CG']*

E6.g.8 Show that if α is algebraic then α^2 is algebraic. *Hidden solution:* *[UNACCESSIBLE UUID '0CJ']* [0CH]

E6.g.9 If α, β are algebraic, prove that $\alpha\beta$ is algebraic. *Hidden solution:* *[UNACCESSIBLE UUID '0CM']* [0CK]

The above shows that algebraic numbers are a field.

§7 Sequences and series [0CN]

§7.a Sequences

Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a real-valued sequence (as defined in 3.e.2).

Given $N \in \mathbb{N}$ we will write $\sup_{n\ge N} a_n$ in the following, instead of $\sup\{a_N, a_{N+1} \dots\}$, and similarly for the infimum. (This is in accordance with 6.c.8)

Exercises

E7.a.1 Prerequisites:6.c.7. [0CP]

We have that $\sup_{n\ge N} a_n = \sigma \in \overline{\mathbb{R}}$ if and only if

$$\forall n \ge N, a_n \le \sigma \quad \text{e} \tag{7.a.2}$$

$$\forall L < \sigma, \exists n \ge N, a_n > L \tag{7.a.3}$$

(note that if $\sigma = \infty$ the first is trivially true, while if $\sigma = -\infty$ the latter is true because there are no L).

Solution. 7.a.4. *It follows from the characterization 6.c.3.* [0CQ]

E7.a.5 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_n \sim n^n$. Prove that, setting $s_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} a_n$ we have $s_n \sim a_n$. [0CR]

E7.a.6 Let e_n, d_n be two real sequences such that $d_n \le e_n$ for each n, and suppose that $\limsup_n e_n = \liminf_n d_n = b$ (possibly infinite): then show that $\lim_n e_n = \lim_n d_n = b$. *Hidden solution:* [UNACCESSIBLE UUID '0CT'] [0CS]

E7.a.7 Prerequisites:6.c.11,6.e.12. Let a_n, b_n real valued sequences, show that [0CV] *(Solved on 2022-11-24)*

$$\limsup_{n\to\infty}(a_n + b_n) \le (\limsup_{n\to\infty} a_n) + (\limsup_{n\to\infty} b_n) \ ;$$

find a case where inequality is strict. *Hidden solution:* [UNACCESSIBLE UUID '0CW']

E7.a.8 Difficulty:*. [0CX]

Let $a_{n,m}$ be a real valued sequence [†52] with two indexes $n, m \in \mathbb{N}$. Suppose that

- for every m the limit $\lim_{n\to\infty} a_{n,m}$ exists, and that
- the limit $\lim_{m\to\infty} a_{n,m} = b_n$ exists uniformly in n and is finite, *i.e.*

$$\forall \varepsilon > 0,\ \exists m \in \mathbb{N}\, \forall n \in \mathbb{N},\ \forall h \ge m\ \ |a_{n,h} - b_n| < \varepsilon\ .$$

then

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} a_{n,m} \tag{7.a.9}$$

in the sense that if one of the two limits exists (possibly infinite), then the other also exists, and they are equal.

Find a simple example where the two limits in (7.a.9) are infinite.

Find an example where $\lim_{m\to\infty} a_{n,m} = b_n$ but the limit is not uniform and the previous equality (7.a.9) does not apply.

Hidden solution: [UNACCESSIBLE UUID '0CZ']

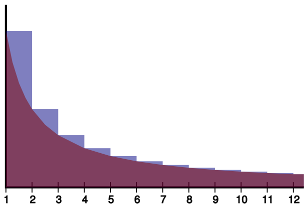


Figure 1: Representation of Euler-Mascheroni constant
Image by William Demchick, Creative Commons Attribution 3.0 Unported License, taken from wikipedia.

E7.a.10 `Prerequisites:7.a.8,7.a.6.`Let again $a_{n,m}$ be a real valued sequence with two indices $n, m \in \mathbb{N}$; suppose that, for every n, the limit $\lim_{m\to\infty} a_{n,m} = b_n$ exists, is finite and is uniform in n; suppose that the limit $\lim_n b_n$ exists and is finite. Can it be concluded that the limits $\lim_{n\to\infty} a_{n,m}$ exist for each fixed m? Can we write an equality as in eqn. (7.a.9) in which, however, on the RHS we use the upper or lower limits of $a_{n,m}$ for $n \to \infty$, instead of the limits $\lim_{n\to\infty} a_{n,m}$? [0D0]

Hidden solution: *[UNACCESSIBLE UUID '0D1']*

E7.a.11 `Difficulty:*.` Show that from any sequence $(a_n)_n$ we can extract a monotonic subsequence. *Hidden solution:* *[UNACCESSIBLE UUID '0D3']* [0D2]

E7.a.12 `Difficulty:*.` Show that from any sequence $(a_n)_n \subseteq \mathbb{R}$ we can extract a monotonic subsequence such that [0D4]

$$\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '0D5']*

E7.a.13 `Topics:Euler-Mascheroni constant.Prerequisites:3.f.4.` [0D6]

Show that the limit

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) \quad .$$

exists and is finite. This γ is called Costante di Eulero - Mascheroni. It can be defined in many different ways (see the previous link) including

$$\gamma = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) \mathrm{d}x$$

where the parentheses $\lfloor\cdot\rfloor$ indicate the floor function $\lfloor x \rfloor \stackrel{\text{def}}{=} \max\{n \in \mathbb{Z} : n \le x\}$. In the image 1 the constant γ is the blue area.

Hidden solution: *[UNACCESSIBLE UUID '0D8']*

E7.a.14 [0D9]

†52 This result applies more generally when $a_{n,m}$ are elements of a metric space; moreover a similar result occurs when the limits $n \to \infty$ and/or $m \to \infty$ are replaced with limits $x \to \hat{x}$ and/or $y \to \hat{y}$ where the above variables move in metric spaces. See for example 18.12.

Let $a_k = \sqrt[3]{k^3+k} - k$. Prove that

$$\sum_{k=1}^{n} a_k \sim \frac{1}{3}\log(n)$$

that is, the ratio between the two above sequences tends to 1 when $n \to \infty$. *Hidden solution:* [UNACCESSIBLE UUID '0DB'][UNACCESSIBLE UUID '0DC']

E7.a.15 Note:*Exercise 1 from the written exam 9 April 2011.*Let (a_n) be a sequence of real numbers, with $a_n \geq 0$. [0DD]

1. Show that if $\sum_{n=1}^{\infty} a_n$ converges then also

$$\sum_{n=1}^{\infty} a_n^2 \quad \text{e} \quad \sum_{n=1}^{\infty}\left(a_n \sum_{m=n+1}^{\infty} a_m\right)$$

converge.

2. Assuming moreover that $\sum_{n=1}^{\infty} a_n$ is convergent, let's define

$$a = \sum_{n=1}^{\infty} a_n \ , \ b = \sum_{n=1}^{\infty}\left(a_n \sum_{m=n+1}^{\infty} a_m\right) \ , \ c = \sum_{n=1}^{\infty} a_n^2$$

then show that $a^2 = 2b + c$.

Exercise 7.a.16. *Let a_n, b_n be real sequences (which can have variable signs, take value zero, and are not necessarily infinitesimal).* [0DJ]

Recall that the notation $a_n = o(b_n)$ means:

$$\forall \varepsilon > 0, \ \exists \overline{n} \in \mathbb{N}, \ \forall n \in \mathbb{N}, n \geq \overline{n} \Rightarrow |a_n| \leq \varepsilon |b_n| \ .$$

Shown that these two clauses are equivalent.

- *Eventually in n we have that $a_n = 0 \iff b_n = 0$; having specified this, we have $\lim_n \frac{a_n}{b_n} = 1$, where it is decided that $0/0 = 1$ (in particular a_n, b_n eventually have the same sign, when they are not both null);*
- *we have that $a_n = b_n + o(b_n)$.*

The second condition appears in Definition 3.2.7 in [2] where it is indicated by the notation $a_n \sim b_n$.

Deduct that $a_n \sim b_n$ is an equivalence relation.

Hidden solution: [UNACCESSIBLE UUID '29Y']

Exercise 7.a.17. *Prerequisites:3.g.3.Let a_n, b_n be real sequences (which can have variable signs, take value zero, and are not necessarily infinitesimal); let $X = \mathbb{R}^{\mathbb{N}}$ the space of all sequences.* [02F]

Recall that the notation $a_n = O(b_n)$ means:

$$\exists M > 0, \ \exists \overline{n} \in \mathbb{N}, \ \forall n \in \mathbb{N}, n \geq \overline{n} \Rightarrow |a_n| \leq M|b_n| \ .$$

Show these results:

- *for $a, b \in X$, $a = (a_n)_n$, $b = (b_n)_n$ consider the relation*

$$aRb \iff a_n = O(b_n)$$

prove that R is a preorder;

- *define $x \asymp y \iff (xRy \wedge yRx)$ then $\asymp$ is an equivalence relation, R is invariant for $\asymp$, and the projection $\preceq$ is an order relation on $X/\asymp$ (hint: use the Prop. 3.g.3).*

- *Define (as usually done)*

$$\hat{a} \prec \hat{b} \iff (\hat{a} \preceq \hat{b} \wedge \hat{a} \neq \hat{b})$$

for $\hat{a}, \hat{b} \in X/\asymp$, $(a_n)_n \in \hat{a}, (b_n)_n \in \hat{b}$ representatives; assuming $b_n \neq 0$ (eventually in n), prove that

$$\hat{a} \prec \hat{b} \iff 0 = \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} < \infty \quad .$$

The above discussion is related to Definition 3.2.3 (and following) in [2].

See also exercises 6.c.11 and 6.c.10.

§7.a.a Summation by parts

Exercises

E7.a.18 Suppose $(a_n)_n, (b_n)_n$ are sequences of real numbers and c_n is defined by 7.c.29; let then [217]

$$A_n = \sum_{h=0}^{n} a_h \ , \ B_n = \sum_{h=0}^{n} b_h \ , \ C_n = \sum_{h=0}^{n} c_h$$

the partial sums of the three series; suppose that $\sum_{n=0}^{\infty} b_n = B$ is convergent: show that

$$C_n = \sum_{i=0}^{n} a_{n-i} B_i = \sum_{i=0}^{n} a_{n-i}(B_i - B) + A_n B \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '216']*

E7.a.19 Note:*Taken from Rudin [22] Prop. 3.41.* [21H]

Let $(a_n)_n (b_n)_n$, be sequences, let $A_n = \sum_{k=0}^{n} a_k$ and $A_{-1} = 0$, $0 \leq p \leq q$, then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \quad .$$

§7.b Recursive sequences

Exercises

E7.b.1 Let $f(x) = x - x^3$ and $x_0 \in \mathbb{R}$, and $(x_n)_{n\in\mathbb{N}}$ a sequence defined by recurrence by $x_{n+1} = f(x_n)$. Prove that there is a $\lambda > 0$ such that if $|x_0| < \lambda$ then $x_n \to 0$, while if $|x_0| > \lambda$ then $|x_n| \to \infty$; and possibly calculate this λ. [0DK]

Hidden solution: [UNACCESSIBLE UUID '0DM']

E7.b.2 Note:*Babylonian method for square root.* Let $S > 0$ and consider the sequence defined by recurrence as [0DN]

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right) \quad ;$$

show that $x_n \to \sqrt{S}$ and that, for $S \in [1/4, 1]$ and $x_0 = 1$, convergence is superquadratic, i.e.

$$\left|x_n - \sqrt{S}\right| \le 2^{1-2^n} \quad .$$

Find a function $f(x)$ (dependent on S) such that the previous iteration can be seen as a Newton's method, i.e.

$$x - \frac{f(x)}{f'(x)} = \frac{1}{2}\left(x + \frac{S}{x}\right).$$

Generalize the Babylonian method to find a root $\sqrt[k]{S}$.

Hidden solution: [UNACCESSIBLE UUID '0DP']

§7.c Series

§7.c.a Tests

Theorem 7.c.1 (Root test). *Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ then* [219]

- *if $\alpha < 1$ the series $\sum_{n=1}^\infty a_n$ converges absolutely;*
- *if $\alpha = 1$ nothing can be concluded;*
- *if $\alpha > 1$ the series $\sum_{n=1}^\infty a_n$ does not converge, and also $\sum_{n=1}^\infty |a_n|$ diverges.*

Proof. [21B]

- If $\alpha < 1$, having fixed $L \in (\alpha, 1)$ you have eventually $\sqrt[n]{|a_n|} < L$ so there is a N for which $|a_n| \le L^{N-n}$ for each $n \ge N$ and we conclude by comparison with the geometric series.
- For the two series $1/n$ and $1/n^2$ you have $\alpha = 1$.
- If $\alpha > 1$ you have frequently $\sqrt[n]{|a_n|} > 1$ So $|a_n| > 1$, contrary to the necessary criterion.

□

Theorem 7.c.2 (Ratio test). *Assume that $a_n \ne 0$. Let $\alpha = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ then* [21C]

- *if* $\alpha < 1$ *the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely;*
- *if* $\alpha \geq 1$ *nothing can be concluded.*

Proof.
- If $\alpha < 1$, taken $L \in (\alpha, 1)$ you have eventually $\frac{|a_{n+1}|}{|a_n|} < L$ so there is a N for which $\frac{|a_{n+1}|}{|a_n|} < L$ for each $n \geq N$, by induction it is shown that $|a_n| \leq L^{n-N}|a_N|$ and ends by comparison with the geometric series.
- Let's see some examples. For the two series $1/n$ and $1/n^2$ you have $\alpha = 1$. Defining

$$a_n = \begin{cases} 2^{-n} & n \text{ even} \\ 2^{2-n} & n \text{ odd} \end{cases} \tag{7.c.3}$$

we obtain a convergent series but for which $\alpha = 2$.

□

Remark 7.c.4. *If the ratio test 7.c.2 can be applied, we have seen in the demonstration that, for a* $L < 1$, *there is a* N *for which* $|a_n| \leq L^{n-N} a_N$ *for every* $n \geq N$, *and therefore* $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq L < 1$, *that is the root test 7.c.5 holds.* [0F1]

Theorem 7.c.5. *If* $(a_n)_n \subset \mathbb{R}$ *has positive terms and is monotonic (weakly) decreasing, the series converges if and only if the series* [21D]

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

converges.

Proof. Since the sequence $(a_n)_n$ is decreasing, then for $h \in \mathbb{N}$

$$2^h a_{2^{(h+1)}} \leq \sum_{k=2^h+1}^{2^{(h+1)}} a_k \leq 2^h a_{2^h} \quad . \tag{7.c.6}$$

We note now that

$$\sum_{h=0}^{N} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \sum_{n=2}^{2^{N+1}} a_n$$

and therefore

$$\sum_{h=0}^{\infty} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \lim_{N\to\infty} \sum_{h=0}^{N} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \lim_{N\to\infty} \sum_{n=2}^{2^{(N+1)}} a_n = \sum_{n=2}^{\infty} a_n \quad .$$

so we can add the terms in (7.c.6) to get

$$\sum_{h=0}^{\infty} 2^h a_{2^{(h+1)}} \leq \sum_{n=2}^{\infty} a_n \leq \sum_{h=0}^{\infty} 2^h a_{2^h}$$

where the term on the right is finite if and only if the one on the left is finite, because

$$\sum_{h=0}^{\infty} 2^h a_{2^h} = a_1 + 2\sum_{h=0}^{\infty} 2^h a_{2^{(h+1)}} \quad ;$$

the proof ends by the comparison theorem □

The Dirichlet criteria implies the Liebniz "alternating series test" criteria.

Theorem 7.c.7 (Dirichlet criterion)**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences. If b_n tends monotonically to 0 and if the series of partial sums of a_n is bounded,* i.e. *if* [21F]

$$b_n \ge b_{n+1} > 0 \quad , \quad \lim_{n\to\infty} b_n = 0 \quad , \quad \exists M > 0,\ \forall N \in \mathbb{N}\,, \left|\sum_{n=1}^{N} a_n\right| < M \quad ,$$

then the series

$$\sum_{n=1}^{+\infty} a_n b_n$$

is convergent.

The proof is left as an exercise (Hint: use 7.a.19)

Hidden solution: [UNACCESSIBLE UUID '21G']

In particular, if we set $a_n = (-1)^n$ we prove the existence of the limit in Leibniz test.

Theorem 7.c.8 (Alternating series test, or Leibniz test)**.** *Let b_n be a sequence for which* [238]

$$b_n \ge b_{n+1} > 0 \quad , \quad \lim_{n\to\infty} b_n = 0 \quad ,$$

then the series

$$\sum_{n=0}^{+\infty} (-1)^n b_n$$

is convergent; also, called ℓ the value of the series, letting

$$B_N = \sum_{n=0}^{N} (-1)^n b_n$$

the partial sums, we have that the sequence B_{2N} is decreasing , the sequence B_{2N+1} is increasing, and both converge to ℓ.

Theorem 7.c.9. *Consider the series $\sum_{n=1}^{\infty} a_n$ where the terms are positive: $a_n > 0$. Define* [0DR] *(Solved on 2022-12-13)*

$$z_n = n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

for convenience.

- *If $z_n \le 1$ eventually in n, then the series does not converge.*
- *If there exists $L > 1$ such that $z_n \ge L$ eventually in n,* i.e. *equivalently if*

  $$\liminf_{n\to\infty} z_n > 1 \quad ,$$

  then the series converges.

In addition, fixed $h \in \mathbb{Z}$, we can define

$$z_n = (n+h)\left(\frac{a_n}{a_{n+1}} - 1\right)$$

or

$$z_n = n\left(\frac{a_{n+h}}{a_{n+h+1}} - 1\right)$$

such as

$$z_n = n\left(\frac{a_{n-1}}{a_n} - 1\right)$$

and the criterion applies in the same way. Hidden solution: [UNACCESSIBLE UUID '0DS']

§7.c.b Exercises

Exercises

E7.c.10 Let $\alpha > 0$; use Raabe's criterion 7.c.9 to study the convergence of the series [214] *(Solved on 2022-12-13)*

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

Hidden solution: *[UNACCESSIBLE UUID '215']*

E7.c.11 Let $\alpha > 0$; use the condensation criterion 7.c.5 to study the convergence of the series [23D] *(Solved on 2022-12-13)*

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

E7.c.12 Given a series $\sum_n^\infty a_n$ tell if the following conditions are necessary and/or sufficient for convergence. [0DW]

$$\forall \varepsilon > 0\ \exists m \in \mathbb{N}\ \forall n > m\ \forall k \in \mathbb{N}\ \left|\sum_{j=n}^{n+k} a_k\right| < \varepsilon \tag{7.c.13}$$

$$\forall \varepsilon > 0\ \forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n > m\ \left|\sum_{j=n}^{n+k} a_k\right| < \varepsilon \tag{7.c.14}$$

$$\forall \varepsilon > 0\ \exists m \in \mathbb{N}\ \forall n > m \forall k \in \mathbb{N}\ \sum_{j=n}^{n+k} |a_k| < \varepsilon \tag{7.c.15}$$

$$\forall \varepsilon > 0\ \forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n > m\ \sum_{j=n}^{n+k} |a_k| < \varepsilon \tag{7.c.16}$$

Hidden solution: *[UNACCESSIBLE UUID '0DX']*

E7.c.17 Find two sequences $(a_n)_n, (b_n)_n$ with $a_n, b_n > 0$ such that $\sum_{n=0}^\infty (-1)^n a_n$ is convergent, $\sum_{n=0}^\infty (-1)^n b_n$ is non-convergent, and $\lim_{n\to\infty} a_n/b_n = 1$. *Hidden solution:* *[UNACCESSIBLE UUID '0DZ']* [0DY] *(Proposed on 2022-12-13)*

E7.c.18 Note:*Exam of 9th APr 2011.*Let (a_n) be a sequence of real numbers (not necessarily positive) such that the series $\sum_{n=1}^\infty a_n$ converges to $a \in \mathbb{R}$; let $b_n = \frac{a_1+\cdots+a_n}{n}$; show that if the series $\sum_{n=1}^\infty b_n$ converges then $a = 0$. [0F0]

E7.c.19 Find two examples of $a_{i,j} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ [0F2] *(Proposed on 2022-12)*

- such that, for each i, $\sum_j a_{i,j} = 0$, while for each j, $\sum_i a_{i,j} = \infty$;
- such that, for each i, $\sum_j a_{i,j} = 0$, while for each j, $\sum_i a_{i,j} = 1$.

Can you find examples where moreover we have that $|a_{i,j}| \le 1$ for every i, j?

E7.c.20 `Note:`*Written exam of 4th Apr 2009, exee 1.* Given a sequence $(a_n)_n$ of strictly positive numbers, it is said that *the infinite product* $\prod_{n=0}^{\infty} a_n$ *converges* if there exists finite and strictly positive the limit of partial products, i.e. [0F4] *(Proposed on 2022-12-13)*

$$\lim_{N\to+\infty} \prod_{n=0}^{N} a_n \in (0, +\infty) \quad .$$

Prove that

1. if $\prod_{n=0}^{\infty} a_n$ converges then $\lim_{n\to+\infty} a_n = 1$;
2. if the series $\sum_{n=0}^{\infty} |a_n - 1|$ converges, then it also converges $\prod_{n=0}^{\infty} a_n$;
3. find an example where the series $\sum_{n=0}^{\infty}(a_n - 1)$ converges but $\prod_{n=0}^{\infty} a_n = 0$.

E7.c.21 We indicate with $\mathcal{P}_f(\mathbb{N})$ the set of subsets $B \subseteq \mathbb{N}$ which are finite sets. This is said *the set of finite parts*. [0F5]

We abbreviate $\mathcal{P} = \mathcal{P}_f(\mathbb{N})$ in the following.

Given a sequence $(a_n)_n$ of real numbers and a $B \in \mathcal{P}$ we indicate with $s(B) = \sum_{n\in B} a_n$ the finite sum with indices in B.

Suppose the series $\sum_{n=0}^{\infty} a_n$ converge but not converge at all. Then:

- $\{s(F) : F \in \mathcal{P}\}$ it is dense in $\mathbb{R}$.
- There is a reordering σ of $\mathbb{N}$, that is, a bijective function $\sigma : \mathbb{N} \to \mathbb{N}$, such that all partial sums $\sum_{n=0}^{N} a_{\sigma(n)}$ (at the variation of N) is dense in $\mathbb{R}$.

E7.c.22 `Note:`*This result is attributed to Riemann , see 3.54 in [22]..* [0F7]

Let be given a sequence $(a_n)_n$ of real numbers such that $\sum_{n=0}^{\infty} a_n$ converges (to a finite value) but $\sum_{n=0}^{\infty} |a_n| = \infty$; for each l, L with $-\infty \le l \le L \le +\infty$ there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that, defining $S_N = \sum_{k=0}^{N} a_{\pi(k)}$, we have that

$$\limsup_{N\to\infty} S_N = L \quad , \quad \liminf_{N\to\infty} S_N = l \quad .$$

E7.c.23 A sequence is given $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$: prove that for every $l \in \mathbb{R}$ there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{1, -1\}$ for each n, such that [0F8]

$$\sum_{n=0}^{\infty} (\varepsilon_n a_n) = l \quad .$$

If instead $\sum_{n=0}^{\infty} a_n = S < \infty$, what can be said about the set E of the sums $\sum_{n=0}^{\infty}(\varepsilon_n a_n) = l$, for all possible choices of $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{1, -1\}$ for every n?

- Analyze cases where $a_n = 2^{-n}$ or $a_n = 3^{-n}$
- Show that E is always closed.
- Under what assumptions do you have that $E = [-S, S]$?

Hint. Let $\tilde{E}$ be the set of sums $\sum_n (\varepsilon_n a_n) = l$, to vary by $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{0,1\}$ for each n; note that $\tilde{E} = \{(S+x)/2 : x \in E\}$.

E7.c.24 Note:*Written exam of 12th Jan 2019.* [0F9]

Show that the following series converges

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}{3 \cdot 6 \cdot 9 \cdot 12 \cdots (3n)} \right)^2$$

Hidden solution: [UNACCESSIBLE UUID '0FB']

E7.c.25 Say for which $\alpha > 0, \beta > 0, \gamma > 0$ you have that [21M] *(Proposed on 2022-12)*

$$\sum_{n=4}^{\infty} \frac{1}{n^\alpha (\log n)^\beta (\log(\log n))^\gamma}$$

converges.

E7.c.26 Note:*Written exam 29th January 2021.*Let it be $\alpha > 0$. Say (justifying) for which α the following series converge or diverge [23F] *(Proposed on 2022-12-13)*

- $$\sum_{n=1}^{\infty} \left(\sqrt[4]{n^8 + n^\alpha} - n^2 \right)$$
- $$\sum_{n=2}^{\infty} \left(\frac{1}{n^\alpha} - \frac{1}{n^\alpha + 1} \right)$$
- $$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{\alpha \log_2(n)}}$$

  where the logarithms are in base 2.

Hidden solution: [UNACCESSIBLE UUID '23G']

E7.c.27 Note:*Task of 26 Jan 2016.* [20Z] *(Solved on 2022-01-20)*

Let

$$z_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)} \quad ;$$

Show that $\lim_{n\to\infty} z_n = 0$ but

$$\sum_{n=1}^{\infty} z_n = \infty \quad .$$

Hidden solution: [UNACCESSIBLE UUID '213']

E7.c.28 Note:*exercise 2, written exam 15 January 2014.* Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers. Having defined $s_n = \sum_{i=0}^{n} a_i$ prove that: [210]

- the series $\sum_{n=0}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} a_n/s_n$ converges;
- the series $\sum_{n=0}^{\infty} a_n/(s_n)^2$ converges.

Hidden solution: *[UNACCESSIBLE UUID '21K']*

See also exercise 24.1.

§7.c.c Cauchy product

Definition 7.c.29. *Give two sequences $(a_n)_n$ and $(b_n)_n$ to real or complex values, their **Cauchy product** is the sequence $(c_n)_n$ given by* [0FH]

$$c_n \stackrel{def}{=} \sum_{k=0}^{n} a_k b_{n-k} \quad .$$

Exercises

E7.c.30 If $\forall n \in \mathbb{N}, a_n, b_n \geq 0$ show that [0FJ]

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n$$

with the convention that $0 \cdot \infty = 0$.

E7.c.31 If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, show that the series $\sum_{n=0}^{\infty} c_n$ converges absolutely and [0FK] *(Proposed on 2022-12-13)*

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n \quad .$$

E7.c.32 Prerequisites:7.a.18.Note:*Known as: Mertens’ theorem..* [0FM]

If the series $\sum_{n=0}^{\infty} a_n$ converges absolutely and $\sum_{n=0}^{\infty} b_n$ converges, show that the series $\sum_{n=0}^{\infty} c_n$ converges and

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '0FN']*

E7.c.33 Discuss the Cauchy product of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ with itself. *Hidden solution:* *[UNACCESSIBLE UUID '0FQ']* [0FP]

See also exercise 19.a.1.

§7.d Generalized sequences, or "nets' [29X]

Definition 7.d.1. *Let in the following* $(J, \leq)$ *be an ordered set with the* filtering property [21J]

$$\forall x, y \in J\ \exists z \in J,\ x < z \wedge y < z \tag{7.d.2}$$

(See section §3.d.a).

A function $f : J \to X$ *is called* **net**.

This f *is a generalization of the concept of* sequence*; indeed the set* $J = \mathbb{N}$ *with its usual ordering has the filtering property*

In this Section we will concentrate on the case $X = \mathbb{R}$.

Remark 7.d.3. *Note that this definition differs from the one generally used; see [42] or [14]; but it is equivalent for all practical purposes, as explained in 3.d.32, 7.d.12, 8.15.* [2B3]

Definition 7.d.4. *Prerequisites:3.d.24, 3.d.28, Sec. §3.d.a.* [0FR]

Given J *a (possibly partially) ordered and filtering set, and given* $f : J \to \mathbb{R}$*, we want to define the concept of limit of* $f(j)$ *"for* $j \to \infty$*".* [†53].

- *We will say that*

$$\lim_{j \in J} f(j) = l \in \mathbb{R}$$

if

$$\forall \varepsilon > 0\ \exists k \in J\ \forall j \in J,\ j \geq k \Rightarrow |l - f(j)| < \varepsilon \quad .$$

Similarly limits are defined $l = \pm\infty$ *(imitating the definitions used when* $J = \mathbb{N}$*.) (This is the definition in the course notes, chap. 4 sect. 2 in [2])*

- *Equivalently we can say that*

$$\lim_{j \in J} f(j) = l \in \overline{\mathbb{R}}$$

if for every neighborhood U *of* l *we have that* $f(j) \in U$ *eventually for* $j \in J$*; where* eventually *has been defined in 3.d.28.*

- *We recall from 3.d.21 that "*a neighborhood of ∞ in J*" is a subset* $U \subseteq J$ *such that* $\exists k \in J \forall j \in J, j \geq k \Rightarrow j \in U$*. Then we can imitate the definition 6.d.1.*

Fixed $l \in \overline{\mathbb{R}}$ *we have* $\lim_{j \in J} f(j) = l$ *when for every "full" neighborhood* V *of* l *in* $\mathbb{R}$*, there exists a neighborhood* U *of* ∞ *in* J *such that* $f(U) \subseteq V$*.*

In particular, this last definition can be used to define the limits of $f : J \to E$ *where* E *is a topological space.*

Definition 7.d.5. *Having fixed* $(a_n)_{n \in \mathbb{N}}$ *a real sequence,* $(a_{n_k})_{k \in \mathbb{N}}$ *is a* subsequence *when* n_k *is a strictly increasing sequence of natural numbers.* [230]

Similarly having fixed $f : J \to \mathbb{R}$*, let* $H \subseteq J$ *be a* cofinal subset *(as defined in 3.d.18): We know from 3.d.25 that* H *is* filtering*. Then the restriction* $h = f_{|H}$ *is a net* $h : H \to \mathbb{R}$*, and is called "*a **subnet** of f*".*

[†53] Note that ∞ is a symbol but it is not an element of J : if it were it should be the maximum, but a filtering set cannot have maximum, *cf* 3.d.24

More in general, suppose that $(H, \le_H)$ is cofinal in $(J, \le)$ by means of a map $i : H \to J$; this means (adapting (3.d.20)) that

$$(\forall h_1, h_2 \in H, h_1 \le_H h_2 \Rightarrow i(h_1) \le i(h_2)) \wedge (\forall j \in J\ \exists h \in H,\ i(h) \ge j) \quad ; \quad (7.d.6)$$

*then $h = f \circ i$ is a **subnet**.*

Exercises

E7.d.7 Prove that the assertions in 7.d.4 are equivalent. [22Z]

E7.d.8 Prerequisites:7.d.4,7.d.1,3.d.26. Show that if the limit $\lim_{j\in J} f(j)$ exists, then it is unique. [0FS]

E7.d.9 Suppose f is monotonic, show that $\lim_{j\in J} f(j)$ exists (possibly infinite) and coincides with $\sup_J f$ (if it is increasing) or with $\inf_J f$ (if it is decreasing). [0FT]

Infer that

$$\limsup_{j\in J} f(j) \stackrel{\text{def}}{=} \lim_{j\in J} \sup_{k\ge j} f(k)$$
$$\liminf_{j\in J} f(j) \stackrel{\text{def}}{=} \lim_{j\in J} \inf_{k\ge j} f(k)$$

are always well defined.

E7.d.10 Show that the limit exists $\lim_{j\in J} f(j) = \ell \in \overline{\mathbb{R}}$ if and only if [0FV]

$$\limsup_{j\in J} f(j) = \liminf_{j\in J} f(j) = \ell\ .$$

E7.d.11 Prerequisites:3.d.13,3.d.25,7.d.4,7.d.1,3.d.27. Suppose $H \subseteq J$ is *cofinal* and let $h = f|_H$ be the subnet (as defined in 7.d.5); [22Y]

Suppose that $\lim_{j\in J} f(j) = l \in \overline{\mathbb{R}}$ show that $\lim_{j\in H} h(j) = l$.

Similarly if $(H, \le_H)$ is cofinal in $(J, \le)$ by means of a map $i : H \to J$, and $h = f \circ i$.

Remark 7.d.12. *Suppose that the set J is directed but not filtering; then by 3.d.24 it admits a maximum element that we call ∞; the above definitions and properties can also be stated in this case, but they are trivial, since* [237]

$$\lim_{j\in J} f(j) = \liminf_{j\in J} f(j) = \limsup_{j\in J} f(j) = f(\infty) \quad .$$

§7.e Generalized series

§7.e.a Generalized series with positive terms

Definition 7.e.1. *Let I be an infinite family of indices and let $a_i : I \to [0, \infty]$ be a generalized sequence, we define the sum $\sum_{i\in I} a_i$ as* [0FW]

$$\sum_{i\in I} a_i = \sup\left\{ \sum_{i\in K} a_i : K \in \mathcal{P}_f(I) \right\}$$

where $\mathcal{P}_f(I)$ is the set of finite subsets $K \subseteq I$.

Exercises

E7.e.2 Prerequisites:3.1.1.Note:*From the written exam of March 27, 2010.*.Say for which $\alpha \in \mathbb{R}$ the series [0FX]

$$\sum_{(m,n)\in\mathbb{N}^2} \frac{1}{(n+m+1)^\alpha} \quad .$$

converges. Then discuss, for $N \ge 3$, the convergence of

$$\sum_{(m_1,\ldots m_N)\in\mathbb{N}^N} \frac{1}{(1+m_1+\ldots+m_N)^\alpha} \quad .$$

Hidden solution: [UNACCESSIBLE UUID '0FY']

E7.e.3 Let I be a family of indices, let a_i be a sequence with $a_i \ge 0$; let moreover $\mathcal{F}$ be a partition of I (not necessarily of finite cardinality); then prove that [0FZ]

$$\sum_{F\in\mathcal{F}} \sum_{i\in F} a_i = \sum_{i\in I} a_i \quad .$$

E7.e.4 Difficulty:*. Let I be a family of indices; let $a_{i,j} : I\times\mathbb{N} \to [0,\infty]$ a generalised succession, such that $j \mapsto a_{i,j}$ is weakly increasing for every fixed i; prove that [0G0]

$$\sum_{i\in I} \lim_{j\to\infty} a_{i,j} = \lim_{j\to\infty} \sum_{i\in I} a_{i,j} \ .$$

(This is a version of the well-known *Monotone convergence theorem*).

Hidden solution: [UNACCESSIBLE UUID '0G2']

E7.e.5 Extend the previous 7.e.4, replacing $\mathbb{N}$ with a set of indexes J endowed with filtering ordering $\le$. [0G3]

§8 Topology [0G5]

Let X be a fixed and non-empty set. We will use this notation. For each set $A \subseteq X$ we define that $A^c = X \setminus A$ is the **complement to A**.

Definition 8.1. *A **topological space** is a pair (X, τ) where X is a non-empty set with associated the family τ of the open sets, which is called **topology**.* [2DY]

Definition 8.2. *A **topology** $\tau \subseteq \mathcal{P}(X)$ is a family of subsets of X that are called **open sets**. This family enjoys three properties: $\emptyset, X$ are open; the intersection of a finite number of open sets is an open sets; the union of an arbitrary number of open sets is an open set.* [0G6]

*A set A is **closed** if A^c is open.*

Definition 8.3. *Let $A, B \subseteq X$ be two subsets.* [0G7]

1. *The **interior** of A, denoted by A°, is the union of all the open sets contained in A, and therefore is the biggest open set contained in A;*
2. *the **closure** of B, denoted by $\overline{B}$, is the intersection of all the closed sets that contain B,* i.e. *is the smallest closed that contains B.*
3. *We say that A is **dense in** B if $\overline{A} \supseteq B$.*[†54]
4. *The **boundary** ∂A of A is $\partial A = \overline{A} \setminus A^\circ$.*

Definition 8.4. *A topological space (X, τ) is said to be T_2, or* "Hausdorff space"*, if $\forall x, y \in X$ exist $U, V \in \tau$ open disjoint with $x \in U, y \in V$.* [0G8]

Definition 8.5. *Any set X can be endowed with many different topologies. Here are two simple examples:* [2F6]

- *When a set X is endowed with the **discrete topology**, then all sets are open, and therefore closed. Equivalently, the discrete topology is caracterized by: every singleton is an open set.*
- *When a set X is endowed with the **indiscrete topology**, then the only open (and, closed) sets are $X, \emptyset$.*

Further informations on these subjects may be found in Chap. 2 of [22] or in [14].

Remark 8.6. *A metric space is a special case of topological space, because the open subsets of the metric space satisfy the Definition 8.2; the associated topology is always Hausdorff. The following results therefore also apply to metric spaces.* [2DH]

Exercises

E8.7 Show that if the space is T_2 then every singleton $\{x\}$ is closed. [0G9]

E8.8 Show that if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$ [0GB]

E8.9 Show that if $A = B^c$ then $(\overline{B})^c = A^\circ$, using the definitions 8.2 and 8.3. [0GC]

E8.10 Note that $A \supseteq A^\circ$ and $B \subseteq \overline{B}$, generally. Show that A is open if and only if $A = A^\circ$; and that B is closed if and only if $B = \overline{B}$, using definitions 8.2 and 8.3. [0GD]

E8.11 `Topics:interior`. Given X, a topological space, and $A \subseteq X$, show that [OGF]

$$A^\circ = (A^\circ)^\circ \ .$$

using the definition of A° given above.

(For the case of X metric space, see also 10.b.16)

Hidden solution: *[UNACCESSIBLE UUID 'OGG']*

E8.12 `Topics:closing`. Given X topological space and $A \subseteq X$, show that [OGH]

$$\overline{A} = \overline{\left(\overline{A}\right)}$$

or by switching to complement with respect to 8.11, and using the definition of $\overline{A}$ like *"intersection of all the locks they contain A"*.

(For the case of X metric space, see also 10.b.19)

E8.13 `Topics:closure, interior`. Let X be a topological space and $A \subseteq X$ open. [OGJ]

1. Show that $A \subseteq \left(\overline{A}\right)^\circ$ (the interior of the closure of A).
2. Find an example of an open set $A \subset \mathbb{R}$ for which $A \neq \left(\overline{A}\right)^\circ$.
3. Then formulate a similar statement for A closed, transitioning on to the complement.

Hidden solution: *[UNACCESSIBLE UUID 'OGK']*

E8.14 Given the sets $A, B \subseteq \mathbb{R}$, determine the relations between the following pairs of sets [OGM]

$$\begin{array}{ccc} \overline{A \cup B} & \text{and} & \overline{A} \cup \overline{B}\,, \\ \overline{A \cap B} & \text{and} & \overline{A} \cap \overline{B}\,, \\ (A \cup B)^\circ & \text{and} & A^\circ \cup B^\circ\,, \\ (A \cap B)^\circ & \text{and} & A^\circ \cap B^\circ\,. \end{array}$$

Hidden solution: *[UNACCESSIBLE UUID 'OGP']*

E8.15 `Prerequisites:3.d.15,3.d.13,3.d.24.Difficulty:*.`*(Replaces 29W)* Let (X, τ) be a topological space. Consider the descending ordering between sets [†55], with this ordering τ is a directed set; we note that it has minimum, given by $\emptyset$. [OGQ]

Now suppose the topology is Hausdorff. Then taken $x \in A$, let $\mathcal{U} = \{A \in \tau : x \in A\}$ be the family of the open sets that contain x: show that $\mathcal{U}$ is a directed set; show that it has minimum if and only if the singleton $\{x\}$ is open (and in this case the minimum is $\{x\}$). [†56]

Hidden solution: *[UNACCESSIBLE UUID 'OGR']*

By the exercise 3.d.24, when $\{x\}$ is not open then $\mathcal{U}$ is a filtering set, and therefore can be used as a family of indices to define a nontrivial "limit" (see Remark 7.d.12). We will see applications in section §8.g.

[†54] Often when you say "A is dense in B" it happens that B is closed and $A \subseteq B$: in this case "dense" is just $\overline{A} = B$.

[†55] To formally reconnect to the definition seen in 3.d.15 we define $A \preceq B \iff A \supseteq B$ and associate the ordering $\preceq$ with τ.

[†56] Note that, the singleton $\{x\}$ is open iff x is an isolated point.

E8.16 Note:*Written exam of 25 March 2017.*Let (X,τ), (Y,θ) be two topological spaces with non-empty intersection and assume that the topologies restricted to $C = X \cap Y$ coincide (i.e. $\tau_{|C} = \theta_{|C}$) [†57] and that C is open in both topologies (i.e. $C \in \tau, C \in \theta$). Prove that there is only one topology σ on $Z = X \cup Y$ such that $\sigma_{|X} = \tau$ and $\sigma_{|Y} = \theta$ and that $X, Y \in \sigma$. *Hidden solution:* *[UNACCESSIBLE UUID 'OGT'][UNACCESSIBLE UUID 'OGV']* [OGS]

§8.a Neighbourhood, adherent point, isolated point, accumulation point [29V]

Definition 8.a.1 (Neighbourhoods). [†58] *Let (X,τ) be a topological space and let $x_0 \in X$.* [OGW]

- *We denote as **neighbourhood** of x_0 any superset of an open set containing x_0 .*
- *We call **fundamental system of neighbourhoods** of x_0 a family $\{U_i\}_{i\in I}$ of neighborhoods x_0 with the property that each neighborhood of x_0 contains at least one of the U_i .*

*We will say that U is an **open neighborhood of** x_0 simply to say that U is an open set that contains x_0.*

Definition 8.a.2. *Let $E, F \subseteq X$ be sets:* [OGX]

- *a point $x_0 \in X$ is an **adherent point of** E if every neighborhood U of x_0 has non-empty intersection with E;*
- *a point $x_0 \in E$ is **isolated in** E if there exists a neighborhood U of x_0 such that $E \cap U = \{x_0\}$;*

(Note that, in some cases, sets can have at most a countable number of isolated points: see 10.g.7 and 8.i.3, and also 10.g.8).

We also define this concept (already seen in 6.b.1 for the case $X = \mathbb{R}$).

Definition 8.a.3 (accumulation point). *Given $A \subseteq X$, a point $x \in X$ is an **accumulation point** for A if, for every neighborhood U of x, $U \cap A \setminus \{x\}$ is not empty.* [†59] [OGY]

*The set of all accumulation points of A is called **derived set** and will be indicated with $D(A)$.*

In the literature *accumulation point* is also called *"limit point"* (which can be confused with the definition 10.b.39); for this reason we will not use this wording. [†60] [2BN]

Exercises

E8.a.4 Check that in the definitions 8.a.2 and 8.a.3 you can equivalently use, instead of the neighborhoods U of x_0, the open neighborhoods U of x_0. [OGZ]

E8.a.5 Check that in the definitions 8.a.2 and 8.a.3 you can equivalently use neighborhoods U of x_0 chosen from a fixed fundamental system of neighborhoods. [OHO]

E8.a.6 Check that the set of points adhering to A coincides with the closure of A. *Hidden solution:* *[UNACCESSIBLE UUID '0H2']* [0H1]

E8.a.7 Prerequisites:8.a.6.Check that $\overline{A} = A \cup D(A)$. *Hidden solution:* *[UNACCESSIBLE UUID '0H4']* [0H3]

E8.a.8 A point $x \in X$ is an accumulation point for X †61 if and only if the singleton $\{x\}$ is not open. *Hidden solution:* *[UNACCESSIBLE UUID '0H6']* [0H5]

E8.a.9 Topics:boundary. Let $A \subset X$. Let's remember the definition of *boundary* $\partial A = \overline{A} \setminus A^\circ$.Note that ∂A is closed: indeed setting $B = A^c$ to be the complement, it is easily verified that $\partial A = \overline{A} \cap \overline{B}$. In particular we showed that $\partial A = \partial B$. [0H7]

Show that the three sets $\partial A, A^\circ, B^\circ$ are disjoint, and that their union is X; in particular, show that the three sets are characterized by these three properties:

- Each neighborhood of x intersects both A and B;
- there exists a neighborhood x contained in A;
- there exists a neighborhood x contained in B.

(See also 10.b.26 for the case of metric spaces). *Hidden solution:* *[UNACCESSIBLE UUID '0H8']*

E8.a.10 Topics:boundary.Difficulty:*. [0H9]

Given X topological space and $A \subseteq X$; if A is open (or closed) the boundary ∂A has empty interior; we have $\partial A \supseteq \partial\partial A$ with equality if ∂A has empty interior; in addition $\partial\partial A = \partial\partial\partial A$. *Hidden solution:* *[UNACCESSIBLE UUID '0HB']* *[UNACCESSIBLE UUID '0HC']*

E8.a.11 Prerequisites:8.a.7.If (X, τ) is a topological space and $A \subset X$ has no isolated points, then also $\overline{A}$ does not have isolated points. *Hidden solution:* *[UNACCESSIBLE UUID '0HF']* [0HD]

E8.a.12 Note:*written exam, 12/1/2013*.Let A be an open subset of X. Prove that, for any subset B of X , the following inclusion holds: $A \cap \overline{B} \subseteq \overline{A \cap B}$. Show, by an example, that the conclusion does not hold if you remove the assumption that A is open. *Hidden solution:* *[UNACCESSIBLE UUID '0HH']* [0HG]

E8.a.13 Given $E \subseteq X$, we distinguish the points $x \in X$ in three distinct sets that are a partition of X. [0HJ]

- For every neighborhood U of x, $U \setminus \{x\}$ intersects E. These are the *accumulation points* of E.
- $x \in E$ and there is a neighborhood U of x such that $U \cap E = \{x\}$. These are the *isolated points* in E.
- *Now describe the third set of points*

Hidden solution: *[UNACCESSIBLE UUID '0HK']*

§8.b Examples [2BD]

Exercises

E8.b.1 Let's consider on $\mathbb{R}$ the family $\tau_+ = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Show that it is a topology. Is it Hausdorff? Calculate closure, interior, boundary and derivative of these sets: [0HM]

$$\{0\} \quad , \quad \{0,1\} \quad , \quad [0,1] \quad , \quad (0,1) \quad ,$$
$$[0,\infty) \quad , \quad (-\infty,0] \quad , \quad (0,\infty) \quad , \quad (-\infty,0) \quad .$$

Hidden solution: [UNACCESSIBLE UUID '0HN']

E8.b.2 Prerequisites:8.h.7,8.h.8.Let $X = \mathbb{R}\cup\{+\infty, -\infty\}$, consider the family $\mathcal{B}$ of parts of X that contains [0HP]

- open intervals (a, b) with $a, b \in \mathbb{R}$ and $a < b$,
- half-lines $(a, +\infty] = (a, +\infty) \cup \{+\infty\}$ with $a \in \mathbb{R}$,
- the half-lines $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$ with $b \in \mathbb{R}$.

(Note the similarity of sets in the second and third points with the "neighbourhoods of infinity" seen in Sec. §6.a).

Show that $\mathcal{B}$ satisfies the properties *(a),(b)* seen in 8.h.7. Let τ therefore be the topology generated from this base. The topological space (X, τ) is called **extended line**, often denoted $\overline{\mathbb{R}}$.

This topological space is T_2, it is compact (Exercise 8.d.6), and is homoemorphic to the interval $[0, 1]$. It can be equipped with a distance that generates the topology described above.

Hidden solution: [UNACCESSIBLE UUID '0HQ']

E8.b.3 Prerequisites:8.h.7,8.h.8.Let $X = \mathbb{R} \cup \{\infty\}$, let's consider the family $\mathcal{B}$ of parts of X comprised of [0HR]

- the open intervals (a, b) with $a, b \in \mathbb{R}$ and $a < b$,
- the sets $(a, +\infty) \cup (-\infty, b) \cup \{\infty\}$ with $a, b \in \mathbb{R}$ and $a < b$.

Show that $\mathcal{B}$ satisfies the properties *(a),(b)* seen in 8.h.7. Let τ therefore be the topology generated by this base.The topological space (X, τ) is called **one-point compactified line**. This topological space is T_2 and it is compact (Exer. 8.d.7); it is homeomorphic to the circle (Exer. 10.o.8); therefore it can be equipped with a distance that generates the topology described above.

E8.b.4 Topics:directed ordering.Prerequisites:3.d.15. [0HS]

†57 Remember that $\tau_{|C} = \{B \cap C : B \in \tau\}$.
†58 Definition 5.6.4 in the notes [2]
†59 We could call $U \setminus \{x\}$ a "deleted neighborhood"; so we are asking that the deleted neighborhood $U \setminus \{x\}$ has non-empty intersection with A; as we already did in 6.b.1.
†60 See in this regard [34].
†61 We are taking $A = X$ in the definition 8.a.3.

Let $(J, \le)$ be a set with direct ordering. We decide that an "open set" in J is a set A that contains a "half-line" of the form $\{k \in J : k \ge j\}$ (for a $j \in J$) [†62]. Let therefore τ be the family of all such open sets, to which we add $\emptyset, J$. Show that τ is a topology. Is this topology Hausdorff? What are the accumulation points?

E8.b.5 Topics:accumulation point, maximum, direct ordering.Prerequisites:3.d.15, 8.b.4. [0HT]

Find a simple example of a set $(J, \le)$ with direct ordering that has maximum but, when we associate to J the topology τ_J of the previous example, (J, τ_J) has no accumulation points.

Hidden solution: *[UNACCESSIBLE UUID '0HV']*

E8.b.6 Topics:direct ordering. Prerequisites:3.d.13, 3.d.15, 3.d.24. [0HW]

Let $(I, \le)$ be a set with direct ordering and with a maximum that we call ∞. We call $J = I \setminus \{\infty\}$ and assume that J is filtering (with induced sorting) and non-empty. In this case we propose a finer topology. The topology τ for I contains:

- $\emptyset, I$;
- sets A that contain a "half-line" $\{k \in I : k \ge j\}$, for a $j < \infty$, (these are called "neighborhoods of ∞");
- subsets of I that do not contain ∞.

Show that τ is a topology. Is this topology Hausdorff? Show that ∞ is the only accumulation point.

Hidden solution: *[UNACCESSIBLE UUID '0HX']*

The previous construction can be used in this way.

Remark 8.b.7. *Let $(J, \le)$ be a non-empty set with filtering order. We know from 3.d.24 that J has no maximum. We extend $(J, \le)$ by adding a point "∞": Let's set $I = J \cup \{\infty\}$ and decide that $x \le \infty$ for every $x \in J$. It is easy to verify that $(I, \le)$ is a direct order, and obviously ∞ is the maximum I.* [†63] *Let τ be the topology defined in 8.b.6. We know that ∞ is an accumulation point. This topology can explain, in a topological sense, the limit already defined in 7.d.4, and other examples that we will see in Sec. §8.g.* [0HY]

§8.c Generated topologies [2BJ]

Exercises

E8.c.1 Prerequisites:3.b.23.Let X be a set and $\mathcal{V} \subseteq \mathcal{P}(X)$ a family of parts of X; we define τ as the intersection of all topologies that contain $\mathcal{V}$ *i.e.* [0J1]

$$\tau \stackrel{\text{def}}{=} \bigcap \{\sigma, \sigma \supseteq \mathcal{V}, \sigma \ \text{ topology in } \ X\}$$

Show that τ is a topology.

τ is the *"topology generated by $\mathcal{V}$"*; it is also called *"the smallest topology that contains $\mathcal{V}$"*.

See also the exercises 8.h.8.

§8.d Compactness [2BF]

Definition 8.d.1. *A subset $K \subseteq X$ is* compact [†64] *if, from every family of open sets $(A_i)_{i \in I}$ whose union $\bigcup_{i \in I} A_i$ covers K, we can choose a finite number $J \subset I$ of open set whose union $\bigcup_{i \in J} A_i$ covers K.* [0J3]

If you formulate these exercises in metric spaces, you can use the theorem 10.j.1 on page 137 to deal with compact sets.

Exercises

E8.d.2 Suppose the topological space is compact. Show that every closed subset is compact. [0J4]

E8.d.3 Suppose the topological space is T_2 (see Definition 8.4). Prove that every compact subset is closed. [0J5]

E8.d.4 `Topics:compact sets.Prerequisites:8.d.3.` `Note:`*For the real case, see 6.c.13. For the case of metric spaces, see 10.j.11..* [0J6]

Let (X, τ) be a T_2 topological space and let $A_n \subseteq X$ be compact nonempty subsets such that $A_{n+1} \subseteq A_n$: then $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

What happens if the space is not T_2? *Hidden solution:* *[UNACCESSIBLE UUID '0J7']*

E8.d.5 `Prerequisites:8.d.3.`Let (X, τ) and (Y, σ) be topological spaces, with X compact and Y T_2. Let $f : X \to Y$ be continuous and injective; show that f is a homeomorphism between X and its image $f(X)$. [0J8]

Hidden solution: *[UNACCESSIBLE UUID '0J9']*

E8.d.6 `Prerequisites:8.b.2.`Show that the *extended line* (the topological space shown in 8.b.2) is compact. *Hidden solution:* *[UNACCESSIBLE UUID '0JC']* [0JB]

E8.d.7 `Prerequisites:8.b.3.`Show that the *compacted line* (the topological space shown in 8.b.3) is compact. [0JD]

See also the exercise 8.f.7 for a characterization of compact sets by *nets*.

§8.e Connection [2BG]

Definition 8.e.1. *Let (X, τ) be a topological space. Given $A, B \subseteq X$, to shorten the formulas we will use the (nonstandard) notation* [2BR]

- *A**i**B to say that A, B have non-empty intersection,*
- *A**d**B to say that they are disjointed, and*
- ***n**A to say that A it is not empty.*

we recall the definition of connectedness *(Chap. 5 Sec. 11 of the notes [2] or, Chap. 2 in [22]).*

- *The space X is disconnected if it is the disjoint union of two open non-empty sets.*

[†62] We could call such a A a *neighborhood of infinity*, as was already done in Sec. §6.a.
[†63] So $(I, \leq)$ is not a filtering order.
[†64] The definition shows that the empty set is compact. Some texts however explicitly exclude this case.

- *The space X is connected if it is not disconnected. This may be rewritten in different fashions, as for example*

$$\forall A, B \in \tau, (\mathbf{n}A \ \wedge \ \mathbf{n}B \ \wedge \ X \subseteq A \cup B) \Rightarrow A\mathbf{i}B \, .$$

- *A non-empty subset $E \subseteq X$is disconnected if it is disconnected with the induced topology; that is, if E is covered by the union of two open sets, each of which intersects E, but which are disjointed in E; in symbols,*

$$\exists A, B \in \tau, E\mathbf{i}A \ \wedge \ E\mathbf{i}B \ \wedge \ E \subseteq A \cup B \ \wedge \ A \cap B \cap E = \emptyset \, . \qquad (8.e.2)$$

- *Similarly a non-empty set $E \subseteq X$ is connected if it is connected with the induced topology. This may be written as*

$$\forall A, B \in \tau, (E\mathbf{i}A \ \wedge \ E\mathbf{i}B \ \wedge \ E \subseteq A \cup B) \Rightarrow A \cap B \cap E \neq \emptyset \, . \qquad (8.e.3)$$

  or equivalently

$$\forall A, B \in \tau, (E \subseteq A \cup B \wedge A \cap B \cap E = \emptyset) \Rightarrow (E \subseteq A \vee E \subseteq B) \, . \qquad (8.e.4)$$

Remark 8.e.5. *It is customary to assume that the empty set is connected; this case, however, is of little interest, generally we will exclude it in the following exercises.* [2BS]

There are many equivalent ways of expressing the above definitions; we leave them as (simple) exercises. This Lemma may also be useful.

Lemma 8.e.6. *If $Y \subseteq X$ is connected and $Y \subseteq E \subseteq \overline{Y}$, then E is connected.* [2FY]
For the proof, See Teorema 5.11.6 in [2], or Theorem 20 in Cap. 1 in [14].

Exercises

E8.e.7 Show that the assertions (8.e.3),(8.e.4) in 8.e.1 are equivalent. *Hidden solution:* [2BT]
[UNACCESSIBLE UUID '2BV']

E8.e.8 The space X is disconnected if and only if it is the disjoint union of two non-empty closed sets. [0JF]

E8.e.9 A non-empty subset $E \subseteq X$ is disconnected if E is covered by the union of two closed sets, each of which intersects E, but which are disjoint inside E. [0JG]

E8.e.10 `Prerequisites:8.e.1.`X is disconnected if and only if there exist non-empty sets $A, B \subseteq X$ whose union covers X, but such that $\overline{B}\mathbf{d}A$ and $B\mathbf{d}\overline{A}$. [0JH]

Hidden solution: *[UNACCESSIBLE UUID '0JJ']*

E8.e.11 `Difficulty:*.`Suppose $E \subseteq X$ is disconnected, can we assume that [0JK]

$$\exists A, B \in \tau, E\mathbf{i}A \ \wedge \ E\mathbf{i}B \ \wedge \ E \subseteq A \cup B \ \wedge \ A\mathbf{d}B \, . \qquad (8.e.12)$$

that is, that there exist two disjoint open sets, each of which intersects E, and that E is covered by their union?

Hidden solution: *[UNACCESSIBLE UUID '0JM'][UNACCESSIBLE UUID '0JP']* See also 10.e.6.

E8.e.13 Let (X, τ_X) be a topological space, $Y \subseteq X$ the topological space with the induced topology [2DK]

$$\tau_Y = \{A \cap Y : A \in \tau_X\} .$$

Fix $E \subseteq Y$, consider these statements.

(cX) E is a connected set in the topological space (X, τ_X);

(cY) E is a connected set in the topological space (Y, τ_Y).

Are the two statements equivalent?

Hidden solution: *[UNACCESSIBLE UUID '113']*

E8.e.14 Note:*Proposition 5.11.2 notes [2].* [2BW]

A set $E \subseteq X$ is disconnected if and only there exists a continuous function $f : E \to \mathbb{R}$ that assumes exactly two values, for example $f(E) = \{0, 1\}$.

E8.e.15 Note:*Theorem 5.11.7 notes [2].* [0JQ]

Let I be a family of indices. Show that if E_i is a family of connected subsets of X such that

$$\forall i, j \in I\ ,\ E_i \cap E_j \neq \emptyset\ ,$$

then $E = \bigcup_{i \in I} E_i$ is connected.

Hidden solution: *[UNACCESSIBLE UUID '0JR']*

Definition 8.e.16. *Given $x \in X$, we will say that the* ***connected component*** *of X containing x is the union of all the connected sets that contain x (note that the singleton $\{x\}$ is connected). The previous exercise 8.e.15 shows that the* connected component *is connected.* [0JT]

Exercises

E8.e.17 Note:*Section 5.11.2 in the text [2].*Show that two connected components are either disjoint or coincide. So the space X is partitioned into connected components. [0JV]

Hidden solution: *[UNACCESSIBLE UUID '0JW']*

E8.e.18 Let $C \subseteq X$ be a closed set; let K be a connected component of C: show that K is closed. *Hidden solution:* *[UNACCESSIBLE UUID '0JZ']* [0JY]

E8.e.19 Find an example of a space (X, τ) where there is a connected component that is not open. *Hidden solution:* *[UNACCESSIBLE UUID '2G0']* [2FZ]

E8.e.20 Let now (X, d) be a metric space where open balls $B(x, r)$ are also closed. Show that the connected components of X are all and only singletons $\{x\}$. [0K0]

(A space where connected sets are always singletons, is called *totally disconnected*).

Hidden solution: *[UNACCESSIBLE UUID '0K1']*

See also the exercises in Sec. §10.e.

§8.f Nets [2B6]

We will use the concepts of *direct order*, *filtering order* and *cofinal set* already discussed in Sec. §3.d.a. In the following (Y, σ) will be a Hausdorff topological space.

Definition 8.f.1. *Let (Y, σ) be a Hausdorff topological space. Let $(J, \le)$ be a set with filtering order (defined in 3.d.13). Let $\varphi : J \to Y$ be a* **net** *(already met in Sec. §7.d).* [0K4]

We define that $\lim_{j \in J} \varphi(x) = \ell \in Y$ if and only if, for every neighborhood V of ℓ in Y you have that $\varphi(j) \in V$ eventually for $j \in J$.

The definition of *eventually* is in 3.d.28, and it means that there exists $k \in J$ such that for every $j \ge k$ you have $\varphi(j) \in V$.

The remark 7.d.3 holds in this case as well.

Definition 8.f.2. *Given a net $x : J \to Y$, a point $z \in Y$ is said to be a* **limit point** *for x if there is a subnet $y : H \to Y$ such that $\lim_{j \in H} y(j) = z$.* [2B4]

(Note that "subnet" is intended in the general sense presented at the end of 7.d.5, where $y = x \circ i$ by means of a map $i : H \to J$ satisfying (7.d.6)).

Exercises

E8.f.3 `Prerequisites:3.d.24.`Let J be a directed but non-filtering set; then let $m \in J$ be its maximum (which exists as seen in 3.d.24); if we define $\lim_{j \in J} \varphi(x)$ as in 8.f.1, show that the limit always exists and it is $\varphi(m)$. [0K5]

E8.f.4 Let (Y, σ) be a Hausdorff topological space and $A \subseteq Y$. Show that $\overline{A}$ coincides with the set of all possible limits of nets $\varphi : J \to A$ (varying J and then φ). [0K6]

E8.f.5 Let (Y, σ) be a Hausdorff topological space and $A \subseteq Y$. Show that $x \in Y$ is an accumulation point for A if and only if there is a J filtering set and there is a net $\varphi : J \to A \setminus \{x\}$ such that $\lim_{j \in J} \varphi(x) = x$. [0K7]

E8.f.6 `Prerequisites:3.d.18,7.d.5.Difficulty:**.` [2B7]

Let (Y, σ) be a Hausdorff topological space. Let J be a filtering set and $x : J \to Y$ be a net in Y. For every $\alpha \in J$ define $E_\alpha \stackrel{\text{def}}{=} \{x_\beta : \beta \in J, \beta \ge \alpha\}$ and

$$E = \bigcap_{\alpha \in J} \overline{E_\alpha}$$

Prove that E coincides with the set L of limit points (defined in 8.f.2).

Hidden solution: *[UNACCESSIBLE UUID '2FK']*

E8.f.7 `Prerequisites:3.d.18,7.d.5,8.f.6.Difficulty:**.` [0K8]

Let (Y, σ) be a Hausdorff topological space. Show that Y is compact if and only if every net taking values in Y admits a converging subnet.

Hidden solution: *[UNACCESSIBLE UUID '0K9']*

§8.g Continuity and limits [2B8]

Definition 8.g.1 (Limit). [†65] *Let (X,τ) and (Y,σ) be two topological spaces, with (Y,σ) Hausdorff.* [†66] *Let $E \subseteq X$ and $f : E \to Y$. Let also x_0 be an accumulation point of E in X. We define that $\lim_{x\to x_0} f(x) = \ell \in Y$ if and only if, for every neighborhood V of ℓ in Y, there exists U neighbourhood of x_0 in X such that $f(U \cap E \setminus \{x_0\}) \subseteq V$.* [0KB]

Definition 8.g.2. *Let (X,τ) and (Y,σ) be two topological spaces, with (Y,σ) Hausdorff; let $f : X \to Y$ be a function.* [2B9]

It is said that f is ***continuous in*** *x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.*

It is said that f is ***continuous*** *if (equivalently)*

- *if f is continuous at every point, that is $\lim_{x\to y} f(x) = f(y)$ for every $y \in X$, or*
- *if $f^{-1}(A) \in \tau$ for each $A \in \sigma$.*

(Thm. 5.7.4 in the notes [2].).

A continuous bijective function $f : X \to Y$ such that the inverse function $f^{-1} : Y \to X$ is again continuous, is called ***homeomorphism****.*

Exercises

E8.g.3 Consider this statement. [2BB]

(Proposed on 2022-12)

«Let $f : X \to Y$ and $x_0 \in X$, then f is continuous at x_0 when, for every open set $B \subseteq Y$ with $f(x_0) \in B$, we have that $f^{-1}(B)$ is open.»

This statement is incorrect.

Build an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at $x_0 = 0$ but such that, for every $J = (a,b)$ open non-empty bounded interval, $f^{-1}(J)$ is not open. *Hidden solution:* *[UNACCESSIBLE UUID '2BC']*

E8.g.4 `Difficulty:*.` [225]

Let Y be a topological space. We say that Y satisfies the property (P) with respect to a topological space X when it satisfies this condition: for every dense subset $A \subseteq X$ and every pair of continuous functions $f, g : X \to Y$ such that $f(a) = g(a)$ for every $a \in A$, necessarily there follows that $f = g$.

Prove that Y is Hausdorff if and only if it satisfies the property (P) with respect to any topological space X.

E8.g.5 `Prerequisites:8.b.7.`Explain how Definition 8.f.1 can be seen as a special case of Definition 8.g.1. (Hint. proceed as in the note 8.b.7 and set $E = J, X = I, x_0 = \infty$). [0KC]

E8.g.6 `Prerequisites:8.g.5.`Let X, Y be topological Hausdorff space. Let $E \subseteq X$, let $f : E \to Y$, and suppose that x_0 is an accumulation point of E in X. [0KD]

- If $\lim_{x\to x_0} f(x) = \ell$ then, for each net $\varphi : J \to X$ with $\lim_{j\in J} \varphi(j) = x_0$ we have $\lim_{j\in J} f(\varphi(j)) = \ell$.

[†65] Definition 5.7.2 in the notes [2].

[†66] To have uniqueness of the limit and therefore to give an unique meaning to $\lim_{x\to x_0} f(x)$ as an element of Y.

- Consider the filtering set J given by the neighborhoods of x_0; [†67] consider nets $\varphi : J \to X$ with the property that $\varphi(U) \in U \setminus \{x_0\}$ for each $U \in J$. We note that $\lim_{j\in J} \varphi(j) = x_0$.
  If for each such net $\lim_{j\in J} f(\varphi(j)) = \ell$, then $\lim_{x\to x_0} f(x) = \ell$.

Hidden solution: [UNACCESSIBLE UUID 'OKF']

E8.g.7 Prerequisites:8.f.3,8.g.5.Let X, Y be Hausdorff topological spaces. Let $f : X \to Y$, $x_0 \in X$. The following are equivalent. [OKG]

1. f is continuous at x_0;
2. for each net $\varphi : J \to X$ such that
$$\lim_{j\in J} \varphi(j) = x_0$$
we have
$$\lim_{j\in J} f(\varphi(j)) = f(x_0) \quad .$$

Hint, for proving that 2 implies 1. Suppose that x_0 is an accumulation point. Consider the filtering set J given by the neighborhoods of x_0; consider nets $\varphi : J \to X$ with the property that $\varphi(U) \in U$ for each $U \in J$; note that $\lim_{j\in J} \varphi(j) = x_0$.

Hidden solution: [UNACCESSIBLE UUID 'OKH']

§8.h Bases [2B5]

Definition 8.h.1 (Base). *Given a topological space (X, τ), a* ***base*** [†68] *is a collection $\mathcal{B}$ of open sets (*i.e.* $\mathcal{B} \subseteq \tau$) with the property that every element of τ is an union of elements of $\mathcal{B}$.* [OKK]

For example, if X is a metric space, then the family of all open balls is a base.

Exercises

E8.h.2 Let $\mathcal{B}$ be a base for a topology τ on X; chosen an open set $A \in \tau$, for every $x \in A$ we can choose a $B_x \in \mathcal{B}$ with $x \in B_x$, and such that $A = \bigcup_{x\in A} B_x$. [OKM]

Hidden solution: [UNACCESSIBLE UUID 'OKN'][UNACCESSIBLE UUID 'OKP']

E8.h.3 Prerequisites:8.h.2.Let $\mathcal{B}$ be a base for a topology τ on X. Show that, given $x \in X$, [OKQ]
$$\{B \in \mathcal{B} : x \in B\}$$
is a fundamental system of neighbourhoods for x .

E8.h.4 Prerequisites:8.a.5, 8.h.3, 8.h.2.Let $\mathcal{B}$ be a base for a topology τ on X. Show that, for any given $A \subseteq X$, [OKS]
$$A^\circ = \bigcup \{B \in \mathcal{B} : B \subseteq A\}$$

[†67] The fact that this is filtering was shown in 3.d.24, 8.15 and 8.a.8

[†68] Also known as *basis*. See [14] page 46, or Chapter 5 Section 6 Definition 5.6.4 in the notes [2], or [49] for an introduction.

while

$$\overline{A} = \{x \in X : \forall B \in \mathcal{B}, x \in B \Rightarrow B \cap A \neq \emptyset\}$$

Hidden solution: *[UNACCESSIBLE UUID 'OKT']*

E8.h.5 Prerequisites:8.h.2. Given X, given a base $\mathcal{C}$ for a topology σ on X, and a base $\mathcal{B}$ for a topology β on X, we have that $\sigma \supseteq \beta$ if and only if for every $x \in X$ and for every $B \in \mathcal{B}, B \ni x$ there exists $C \in \mathcal{C}, C \ni x, C \subseteq B$. *Hidden solution:* *[UNACCESSIBLE UUID 'OM8']* [OM7]

E8.h.6 Prerequisites:8.c.1. Let $X = \{1,2,3\}$ and let $\mathcal{B} = \{\{1,2\},\{2,3\}\}$; let τ be the smallest topology that contains $\mathcal{B}$, show that $\mathcal{B}$ is not a base for τ. [OKV]

Hidden solution: *[UNACCESSIBLE UUID 'OKW']*

It is therefore interesting to try to understand when a family $\mathcal{B}$ can be the base for a topology.

E8.h.7 Let $\mathcal{B}$ be a base for a topology τ on X; then the following two properties apply. [OKX]

(a) $\bigcup \mathcal{B} = X$ that is, the union of all the elements of the base is X.

(b) Given $B_1, B_2 \in \mathcal{B}$ for each $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Hidden solution: *[UNACCESSIBLE UUID 'OKY']*

E8.h.8 Prerequisites:8.c.1,ZF:4. Conversely, let X be a set and $\mathcal{B}$ a family of subsets that verify the previous properties *(a),(b)* seen in 8.h.7. Let σ the family of sets that are obtained as a union of elements of $\mathcal{B}$, in symbols †69 [OKZ]

$$\sigma \stackrel{\text{def}}{=} \left\{ \bigcup_{i \in I} A_i \,:\, I \text{ family of indexes and } A_i \in \mathcal{B} \forall i \in I \right\} ;$$

it is meant that also $\emptyset \in \sigma$. Show that σ is a topology.

Hidden solution: *[UNACCESSIBLE UUID 'OM0']*

E8.h.9 Prerequisites:Generated topology 8.c.1, 8.h.7, 8.h.8. Let's resume 8.h.8. Let again X be a set and $\mathcal{B}$ a family of subsets that satisfy the above properties *(a),(b)* seen in 8.h.7; suppose τ the smallest topology that contains $\mathcal{B}$. Prove that $\mathcal{B}$ is a base for τ. [OM1]

Hidden solution: *[UNACCESSIBLE UUID 'OM2']*

We can therefore say that a family that satisfies *(a),(b)* is a base for the topology it generates. This answers the question posed in 8.h.6.

E8.h.10 Prerequisites:8.c.1,8.h.7,8.h.8. Let now $X_1, \ldots X_n$ be topological spaces with topologies, respectively, $\tau_1, \ldots \tau_n$; let $X = \prod_{i=1}^n X_i$ be the Cartesian product. We apply the above results to define the **product topology** τ: this can be described in two equivalent ways. [OM3]

†69 As already discussed in ZF:4, you could also use the more compact notation $\sigma \stackrel{\text{def}}{=} \left\{ \bigcup \mathcal{F} : \mathcal{F} \subseteq \mathcal{B} \right\}$.

- Union of all Cartesian products of open sets [†70]

$$\tau = \Big\{ \bigcup_{j\in J} \prod_{i=1}^{n} A_{i,j} \,:\, A_{1,j}\in\tau_1,\dots A_{n,j}\in\tau_n \forall j\in J, J \text{ arbitrarily chosen sets of indexes} \Big\} \ .$$

- τ is the smallest topology that contains Cartesian products of open sets.

Hidden solution: [UNACCESSIBLE UUID '0M4']

E8.h.11 Prerequisites:8.h.10,8.h.7,8.h.8.Let now $X_1,\dots X_n$ be topological spaces with topologies $\tau_1,\dots\tau_n$ respectively and suppose that $\mathcal{B}_1,\mathcal{B}_2,\dots\mathcal{B}_n$ are bases for these spaces. Let $X=\prod_{i=1}^n X_i$ be the Cartesian product, and let [0M5]

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} A_i \,:\, A_1\in\mathcal{B}_1, A_2\in\mathcal{B}_2,\dots A_n\in\mathcal{B}_n \right\}$$

The family of all cartesian products of elements chosen from their respective bases. Show that $\mathcal{B}$ is a base for the product topology. (This exercise generalizes the previous 8.h.10, taking $\mathcal{B}_i=\tau_i$).

Hidden solution: [UNACCESSIBLE UUID '0M6']

See also the exercise 10.b.33 for an application to the case of metric spaces.

E8.h.12 Prerequisites:8.h.10,8.h.11,8.h.5. Let, more in general, I be a non-empty index set, and let (X_i,τ_i) be topological spaces, for $i\in I$; let $\mathcal{B}_i$ be a base for τ_i. (Note that the choice $\mathcal{B}_i=\tau_i$ is allowed.) [2F7]

Let $X=\prod_{i\in I} X_i$ be the Cartesian product.

We define the *product topology* τ on X, similarly to 8.h.10, but with a twist.

A base $\mathcal{B}$ for τ is the family of all sets of the form $A=\prod_{i\in I} A_i$ where

$$\forall i\in I, A_i\in\mathcal{B}_i \vee A_i = X_i \ ,$$

and moreover $A_i=X_i$ but for finitely many i.

Show that $\mathcal{B}$ satisfies the requirements in 8.h.7, so it is a base for the topology τ that it generates. Show that the product topology does not depend on the choice of the bases $\mathcal{B}_i$. *Hidden solution:* [UNACCESSIBLE UUID '2F8']

E8.h.13 Prerequisites:3.d.15, 8.h.7. We verify that what is expressed in 8.15 also applies to the "base". Let $\mathcal{B}$ be a base for a topology τ on X; consider the descending order between sets (formally $A\preceq B \iff A\supseteq B$); with this order $(\mathcal{B},\preceq)$ is a directed set, whose minimum is $\emptyset$. Now suppose the topology is Hausdorff. Then taken $x\in X$, let $\mathcal{U}=\{A\in\mathcal{B} : x\in A\}$ be the family of elements of the base that contain x: show that $\mathcal{U}$ is a directed set. Show that it has minimum if and only if the singleton $\{x\}$ is open. *Hidden solution:* [UNACCESSIBLE UUID '0MB'] [0M9]

E8.h.14 Consider a totally ordered set X (that has at least two elements), and the family [2F5]

[†70] As defined at the beginning of section 6, chapter 5, of the notes [2].

$\mathcal{F}$ of all open-ended intervals

$$(x,\infty) \stackrel{\text{def}}{=} \{z \in X : x < z\} \ , \ (-\infty, y) \stackrel{\text{def}}{=} \{z \in X : z < y\} \ , \\ (x,y) \stackrel{\text{def}}{=} \{z \in X : x < z < y\} \qquad (8.h.15)$$

for all $x, y \in X$. (*Cf.* 3.d.45.) Prove that this is a base for a topology, *i.e.* that it satisfies 8.h.8. So $\mathcal{F}$ is a base for the topology τ that it generates. This topology τ is called **order topology**.

If X has no maximum and no minimum, then only the intervals (x, y) are needed to form a base for τ. This is the case for the standard topologies on $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$,

E8.h.16 `Prerequisites:8.h.12.`Consider topological spaces (X_i, τ_i), each with the discrete topology (and each X_i has at least two elements). Let $I = \mathbb{N}$ or $I = \{0, 1, \dots N\}$; let $X = \prod_{i\in I} X_i$ be the Cartesian product. We define the *product topology* τ on X, as explained in 8.h.12. Describe a simple base for this topology. Moreover, if $I = \mathbb{N}$, show that the topology τ is not the discrete topology. [2FD]

Hidden solution: [UNACCESSIBLE UUID '2FF']

E8.h.17 `Prerequisites:8.h.14,8.h.12,3.d.34,8.h.12.` [2F9]

Consider totally ordered sets $(X_i, \preceq_i)$ (each has at least two elements), and the associated *order topologies* τ_i.

Let $I = \mathbb{N}$ or $I = \{0, 1, \dots N\}$; let $X = \prod_{i\in I} X_i$ be the Cartesian product.

Consider these two topologies.

- We define the *product topology* τ on X, as explained in 8.h.12.
- We order X with the lexicographical order $\preceq$, and then we build the order topology σ on X. (See 3.d.34,8.h.12)

Is there an inclusion between σ and τ?

If every X_i is finite, prove that these two topologies coincide [†71].

Hidden solution: [UNACCESSIBLE UUID '2FC']

§8.i First- and second-countable spaces [2BK]

Definition 8.i.1. *A topological space satisfies the first axiom of countability if each point admits a fundamental system of neighborhoods that is countable.* [0MC]

Definition 8.i.2. *A topological space satisfies the second axiom of countability when it has a countable base.* [0MD]

Exercises

E8.i.3 `Difficulty:*.`If (X, τ) satisfies the second axiom of countability, if $A \subseteq X$ is composed only of isolated points, then A has countable cardinality. *Hidden solution:* [UNACCESSIBLE UUID '0MG'] [0MF]

E8.i.4 Prerequisites:8.h.4. If (X,τ) satisfies the second axiom of countability, given $A \subseteq X$ there exists a countable subset $B \subseteq A$ such that $\overline{B} \supseteq A$. In particular, the whole space X admits a dense countable subset: X is said to be *separable*. The vice versa holds for example in metric spaces, see 10.b.28. See also 10.g.3 for an application in $\mathbb{R}^n$. [0MH]

Hidden solution: [UNACCESSIBLE UUID '0MJ']

The countability axioms will return in exercises 10.b.27 and 10.b.28.

§8.j Non-first-countable spaces [2BM]

Exercises

E8.j.1 [†72] Prerequisites:8.h.12,8.h.1.Difficulty:*.Let Ω be a non-empty set; let's consider $X = \mathbb{R}^\Omega$. [0MM]

1. Let
$$U^f_{E,\rho} = \{g \in X, \forall x \in E, |f(x) - g(x)| < \rho\}$$
where $f \in X$, $\rho > 0$ and $E \subset \Omega$ is finite. Show that the family of these $U^f_{E,\rho}$ satisfies the requirements of 8.h.8, and is therefore a *base* for a topology τ (Hint: use 8.h.12). This topology is the *product topology* of topologies of $\mathbb{R}$.
In particular for each $f \in X$ the sets $U^f_{E,\rho}$ are a fundamental system of neighborhoods.
2. Check that the topology is T_2.
3. Note that X is a vector space, and show that the "sum" operation is continuous, as an operation $X \times X \to X$; to this end, show that if $f, g \in X, h = f + g$, for every neighborhood V_h of h there are neighborhoods V_f, V_g of f, g such that $V_f + V_g \subseteq V_h$.
4. Given $B_i \subset \mathbb{R}$ open and non-empty, one for each $i \in \Omega$, show that $\prod_i B_i$ is open if and only if $B_i = \mathbb{R}$ except at most finitely many i.

Hidden solution: [UNACCESSIBLE UUID '0MN']

E8.j.2 Prerequisites:8.h.1,8.j.1,8.f.1.Difficulty:*.Let Ω be an infinite uncountable set ; consider $X = \mathbb{R}^\Omega$ with the topology τ seen in 8.j.1. [2BP]

1. Show that every point in (X,τ) does not admit a countable fundamental system of neighborhoods.
2. Setting
$$C \stackrel{\text{def}}{=} \{f \in X, f(x) \neq 0 \text{ for at most countably many } x \in \Omega\} \qquad (8.j.3)$$
show that $\overline{C} = X$;
3. and that if $(f_n) \subset C$ and $f_n \to f$ pointwise then $f \in C$.

[†71]Note that the order topology on a finite set is also the discrete topology; use 8.h.16.
[†72]These two exercises 8.j.1,8.j.2, are taken from a text originally published by Prof. Ricci in `http://dida.sns.it/dida2/cl/08-09/folde0/pdf9` in March 2014.

4. Let I be the set of all finite subsets of Ω, this is a filtering set if sorted by inclusion; consider the net

$$\varphi : I \to X \quad , \varphi(A) = \mathbb{1}_A$$

then $\forall A \in I, \varphi(A) \in C$ but

$$\lim_{A \in I} \varphi(A) = \mathbb{1}_X \notin C \quad .$$

Hidden solution: [UNACCESSIBLE UUID '2BQ']

E8.j.4 Difficulty:*. We restrict the topology described in the previous example to the set $Y = [0,1]^{[0,1]}$ (that is, we restrict $\mathbb{R}$ to $[0,1]$, and set $\Omega = [0,1]$). Find a sequence $(f_n) \subset Y$ that does not allow a convergent subsequence. [0MP]

Hidden solution: [UNACCESSIBLE UUID '0MQ']

Let's recall the definition 8.d.1: a space X is "compact by coverings" *if, for every $(A_i)_{i \in I}$ family of open such that $\bigcup_{i \in I} A_i = X$, there is a finite subfamily $J \subset I$ such that $\bigcup_{i \in J} A_i = X$. The* Tychonoff theorem *shows that this space Y is "compact by coverings". This exercise shows you instead that Y it is not "compact by sequences".*

§9 Miscellanea [2FB]

This Section hosts material that would not otherwise fit properly elsewhere.

§9.a Polygons [2G3]

We present some simple geometrical properties of polygons, that may be rigorously proven either by analytical methods (embedding geometrical objects as subsets of the Cartesian plane), or purely geometrical methods (in the spirit of [11]).

In the following we will use the celebrated Jordan Theorem; a simple proof may be found in [26].

Theorem 9.a.1. *Let $\varphi : [0,1] \to \mathbb{R}^2$ be simple closed curve in the plane and $C = \varphi([0,1])$ be its trace. (See 21.a.1 for the definition). The complement $\mathbb{R}^2 \setminus C$ consists of exactly two connected components, that are open. One of these components is bounded (and is called* "the interior of the curve", *or,* "the region bounded by the curve"*) and the other is unbounded (the exterior). The curve C is the boundary of each component.* [2FW]

The proof of the Jordan Theorem usually starts with a simple Lemma (again, see [26]; or Theorem 6 [11]).

Definition 9.a.2. *By* polygonal curve $\varphi : [0,1] \to \mathbb{R}^2$ *we will mean: a not self-intersecting (that is, injective) polygonal (that is, piecewise linear) curve in the plane. Analytically, there are points $V_0, V_1, \dots V_n$ (called "vertices") in the plane, and $0 = t_0 < t_1 \dots < t_n = 1$ such that* [2G6]

$$\varphi(t) = \frac{t - t_i}{t_{i+1} - t_i} V_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} V_i \text{ when } t_i \le t \le t_{i+1} .$$

The polygonal curve *is closed if $\varphi(0) = \varphi(1)$. (In this case we require that φ is injective when restricted to $[0,1)$).*

Lemma 9.a.3. *Let $C = \varphi([0,1])$, let P be the region bounded by the closed polygonal curve, and E the exterior; recall that C, P, E is a partion of the plane. Choose $A, B \notin C$ and suppose that the segment AB meets C in k points, none a vertex. Then: if k is odd, A, B are in different regions, $A \in P \Leftrightarrow B \notin P$; if k is even, A, B are in the same region, $A \in P \Leftrightarrow B \in P$.* [2FX]

Definition 9.a.4. *A* polygon *is the plane figure bounded by the polygonal closed curve.* †73 [2FN]

Remark 9.a.5. *Consider a polygonal curve, with n vertexes labeled $V_1, \dots V_n$; this bounds a polygon: how many sides does it have?* [2CD]

It depends. If some vertexes (in sequence) are aligned, then the figure in the plane will visually have less than n sides and vertexes. For this reason, we will distinguish the unlabeled polygon *(which is the subset of the plane) from the* labeled polygon *(in which we also take into account the position of the vertexes); the latter is less intuitive, but makes for better mathematics. See figure 3 on page 121.*

Figure 2: Examples of polygons with many sides (odd or even) and only two ears. Figure for 9.a.8

Exercises

E9.a.6 Difficulty:∗.Let $n \geq 3$ integer; consider a polygon of $n+1$ vertices. Show that it can be cut in two polygons, one with h and one with k sides, and $3 \leq h \leq n$, $3 \leq k \leq n$. By "cut" we mean, two vertexes of the polygon (not contiguous) can be connected by a line that is internal and does not touch other vertexes or sides. The intersection of the two polygons is the segment BD, they do not have other points in common. [29Z]

*Hint. there is at least one vertex B "convex" in which the inner angle β is "convex" (*i.e. *$0 < \beta \leq \pi$ radians); call A, C the vertexes contiguous to B; reason on the triangle ABC.*

Hidden solution: *[UNACCESSIBLE UUID '1QT']*

E9.a.7 Prove by induction that the sum of the internal angles of a polygon with $n \geq 3$ sides, is $(n-2)\pi$. [1XH]

(The proof is easy if the polygon is convex; in the general case 9.a.8 can be useful).

Hidden solution: *[UNACCESSIBLE UUID '1XM']*

E9.a.8 An *ear* of a polygon is the triangle ABC formed by three consecutive vertices A, B, C of the polygon, such that the segment AC lies inside the polygon. This implies that the triangle ABC does not contain any point of the polygonal curve in its interior; and that the two segments AB, BC can be removed from the polygon and replaced with AC to create a newer polygon. Two ears are *non-overlapping* if their interiors do not intersect, or equivalently if the do not have a side in common. [0JN]

Prove the Two ears theorem: every polygon with more than three vertices has at least two non-overlapping ears. (See [17, 33] for more details).

(Hint: consider labelled polygons, to avoid the complication presented in figure 3 on the following page.)

Hidden solution: *[UNACCESSIBLE UUID '2FV']*

E9.a.9 Difficulty:∗. Show that each polygon can be *"triangulated"*, *i.e.* decomposed as a union of nonoverlapping triangles. [†74] [1XW]

Hidden solution: *[UNACCESSIBLE UUID '1XX']*

E9.a.10 Again, we say that a vertex B is "convex" if the inner angle β is "convex" (*i.e.* $0 < \beta \leq \pi$ radians). Prove that the polygon is convex if and only if all its vertices are convex. *Hidden solution:* *[UNACCESSIBLE UUID '2G5']* [2FP]

[†73] The polygonal curve is part of the polygon. Other definitions are possible. See [64].
[†74] It is legitimate if two different triangles have an edge or a vertex in common.

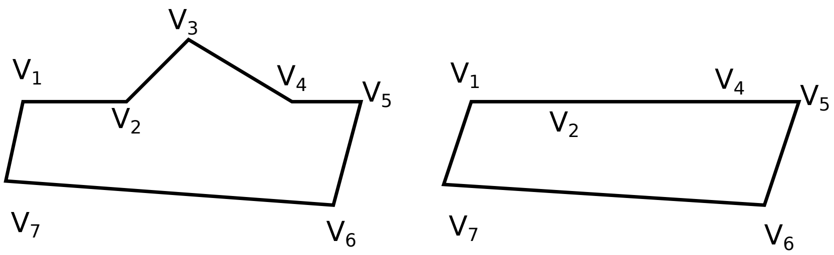


Figure 3: A polygon where, removing an ear, the number of unlabelled sides drops from 7 to 4.

§9.b Cantor set

Let in the following $C \subset \mathbb{R}$ be Cantor's ternary set. This set is described in many texts, as for example Sect. 2.44 in [22]; and also in Wikipedia [51]).

Exercises

E9.b.1 *(Replaces 0W4)* Show that C is closed, and composed only of accumulation points. Hence C is a *perfect set*. [09S]

E9.b.2 Let $I = \{0, 2\}$ and $X = I^{\mathbb{N}}$, consider the map $F : X \to C$ given by [09T]

$$F(x) = \sum_{n=0}^{\infty} 3^{-n-1} x_n \ .$$

Show that it is a bijection.

Let's now equip X with the topology defined in 8.h.17. [†75]. Show that F is a homeomorphism.

Hidden solution: *[UNACCESSIBLE UUID '09V']*

See also 11.22, 10.m.16, 10.b.42.

[†75]Note that the order topology on $I = \{0, 2\}$ is also the discrete topology.

§10 Metric spaces [OMR]

§10.a Definitions [2CC]

A metric space is a pair (X, d) where X is a set (nonempty) with associated distance d.

Definition 10.a.1. *A **distance** is a function $d : X \times X \to [0, \infty)$ that enjoys the following properties:* [OMS]

- $d(x, x) = 0$;
- *(separation property) if* $d(x, y) = 0$ *then* $x = y$;
- *(symmetry)* $d(x, y) = d(y, x)$ *for each* $x, y \in X$;
- *(triangle inequality)* $d(x, z) \le d(x, y) + d(y, z)$ *for each* $x, y, z \in X$.

An example is $\mathbb{R}^n$ with the Euclidean distance $d(x, y) = |x - y|$.

Definition 10.a.2. *Given a sequence* $(x_n)_n \subseteq X$ *and* $x \in X$, [OMT]

- *we will say that "$(x_n)_n$ **converges to** x" if* $\lim_n d(x_n, x) = 0$; *we will also write* $x_n \to_n x$ *to indicate that the sequence converges to* x.
- *We will say that "$(x_n)_n$ **is a Cauchy sequence**" if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}\,,\ \forall n, m \ge N \ \ d(x_n, x_m) < \varepsilon \ .$$

Example 10.a.3. *To any given set* X *we may associate the **discrete distance*** [2C1]

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \ne y \end{cases}$$

The induced topology is the discrete topology *where every subset of* X *is an open set.*

[Note. If you are not familiar with the concept of metric space, you can assume that $X = \mathbb{R}^n$ *and* $d(x, y) = |x - y|$ *in all exercises.]*

Exercises

E10.a.4 Prove that a converging sequence $(x_n)_n \subseteq X$ is Cauchy. [OMV]

E10.a.5 Given a sequence $(x_n)_n \subseteq X$ show that, if it converges to x and converges to y, then $x = y$. [OMW]

This result is known as *Theorem of the uniqueness of the limit*.

E10.a.6 We generalize the definition of *metric space* assuming that $d : X \to [0, \infty]$ (the other axioms are the same). Show that the relation $x \sim y$ defined by [OMX]

$$x \sim y \iff d(x, y) < \infty$$

is an equivalence relation, and that equivalence classes are open, and therefore are disconnected from each other.

Hidden solution: [UNACCESSIBLE UUID 'OMY']

E10.a.7 Given f, g continuous functions on $\mathbb{R}$, we define [OMZ]

$$d(f,g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)| \ .$$

Prove that d is a distance on $X = C(\mathbb{R})$, in the extended sense of the exercise 10.a.6.

Let $f \sim g \iff d(f,g) < \infty$ as before, show that the family of equivalence classes $X/\sim$ has the cardinality of the continuum.

Hidden solution: *[UNACCESSIBLE UUID 'ON0']*

E10.a.8 Prerequisites:10.b.21.Note:*See also eserc. 15.d.2.* Suppose $\varphi : [0,\infty) \to [0,\infty)$ is monotonic weakly increasing and subadditive, *i.e.* $\varphi(t) + \varphi(s) \ge \varphi(t+s)$ for each $t, s \ge 0$; and suppose that $\varphi(x) = 0$ if and only if $x = 0$. [ON1]

Then $\varphi \circ d$ is again a distance. Examples: $\varphi(t) = \sqrt{t}$, $\varphi(t) = t/(1+t)$, $\varphi(t) = \arctan(t)$, $\varphi(t) = \min\{t, 1\}$.

Moreover show that if φ is continuous in zero then the associated topology is the same. †76 *Hidden solution:* *[UNACCESSIBLE UUID 'ON2']*

E10.a.9 If $(x_n)_n \subset X$ is a sequence and $x \in X$, show that $\lim_{n\to\infty} x_n = x$ if and only if, for each sub–sequence n_k there exists a sub–sub–sequence n_{k_h} such that $\lim_{h\to\infty} x_{n_{k_h}} = x$. *Hidden solution:* *[UNACCESSIBLE UUID 'ON4']* [ON3]

E10.a.10 A sequence $(x_n) \subset X$ is a Cauchy sequence if and only if [ON5]

$$\lim_{N\to\infty} \sup\{d(x_n, x_m) : n \ge N, m \ge N\} = 0 \ .$$

E10.a.11 A sequence $(x_n) \subset X$ is a Cauchy sequence if and only if there exists a sequence ε_n with $\varepsilon_n \ge 0$ and $\varepsilon_n \to_n 0$ such that, for every n and every $m \ge n$, we have $d(x_n, x_m) \le \varepsilon_n$. [ON6]

Hidden solution: *[UNACCESSIBLE UUID 'ON7']*

E10.a.12 If $(x_n) \subset X$ is a Cauchy sequence and there exists x and a subsequence n_m such that $\lim_{m\to\infty} x_{n_m} = x$ then $\lim_{n\to\infty} x_n = x$. [ON8]

Hidden solution: *[UNACCESSIBLE UUID 'ON9']*

This "lemma" is used in some important proofs, e.g. to show that a compact metric space is also complete.

E10.a.13 Let $\varepsilon_n > 0$ be an infinitesimal decreasing sequence. If $(x_n) \subset X$ is a Cauchy sequence, there exists a subsequence n_k such that [ONC]

$$\forall k \in \mathbb{N},\ \forall h \in \mathbb{N},\ h > k \Rightarrow d(x_{n_k}, x_{n_h}) \le \varepsilon_k \ .$$

Hidden solution: *[UNACCESSIBLE UUID 'OND']* This property is often used by choosing $\varepsilon_n = 2^{-n}$, or other sequence whose series converges.

E10.a.14 Let $(x_n)_n$ be a sequence such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$: prove that it is a Cauchy sequence. [ONF]

Compare this exercise, the previous 10.a.13 in case $\sum_n \varepsilon_n < \infty$, and exercise 10.a.12.

E10.a.15 If $(x_n) \subset X$ is a Cauchy sequence, $(y_n) \subset X$ is another sequence, and $d(x_n, y_n) \to_n 0$, then $(y_n) \subset X$ is a Cauchy sequence. [ONG]

E10.a.16 Given (X, d) a metric space, show that d is continuous (as a function $d : X \times X \to \mathbb{R}$). You can actually show that it is Lipschitz, by associating to $X \times X$ the distance [ONH]

$$\hat{d}(x, y) = d(x_1, y_1) + d(x_2, y_2), \text{ for } x = (x_1, x_2), y = (y_1, y_2) \in X \times X .$$

Hidden solution: [UNACCESSIBLE UUID 'ONK']

E10.a.17 Prerequisites:6.c.11, 15.d.2,10.a.11.Difficulty:*.Note:*Exercise 2, written exam, 9 July 2011.* [ONM]

Let $\alpha(x)$ be a continuous function on $\mathbb{R}$, bounded and strictly positive. Given f, g continuous functions on $\mathbb{R}$, we define

$$d(f, g) = \sup_{x\in\mathbb{R}} \left(\min\{\alpha(x), |f(x) - g(x)|\} \right) .$$

Prove that d is a distance on $C(\mathbb{R})$ and that $\big(C(\mathbb{R}), d\big)$ is complete. *Hidden solution:* [UNACCESSIBLE UUID 'ONP']

E10.a.18 Note:*Exercise 2, written exam, 25 March 2017.* [ONQ]

Show that the following properties are equivalent for a metric space X:

- every sequence of elements of X admits a Cauchy subsequence;
- The completion X^* of X is compact.

Hidden solution: [UNACCESSIBLE UUID 'ONR']

§10.b Topology in metric spaces [2C2]

Let (X, d) be a metric space.

Definition 10.b.1 (ball,disc)**.** *Let $x \in X, r > 0$ be given; we will indicate with $B(x, r)$ the **ball**,* [ONW]

$$B(x, r) \overset{def}{=} \{y \in X : d(x, y) < r\}$$

that is also indicated with $B_r(x)$; and with

$$D(x, r) \overset{def}{=} \{y \in X : d(x, y) \le r\}$$

*the **disk**, that is also indicated with $\overline{B}_r(x)$.*

Definition 10.b.2. *For the following exercises we define that* [ONX]

1. *a set E is **open** if*

$$\forall x_0 \in E, \exists r > 0 : B(x_0, r) \subseteq E \quad . \tag{10.b.3}$$

It is easily seen that $\emptyset, X$ are open sets; that the intersection of a finite number of open sets is an open set; that the union of an arbitrary number of open set is an open set. So these open sets form a topology.

[†76] See Sec. §10.b below for a summary of definitions regarding topology in metric spaces: in particular the result 10.b.21 will be useful.

2. *The **interior** E° of a set E is*
$$E^\circ = \left\{x \in E : \exists r > 0, B_r(x) \subseteq E\right\} ; \qquad (10.b.4)$$
It is easy to verify that $E^\circ \subseteq E$, and that E is open if and only if $E^\circ = E$ (exercise 10.b.13).

3. *A set is **closed** if the complement is open.*

4. *A point $x_0 \in X$ is **adherent** to E if*
$$\forall r > 0\,, \quad E \cap B_r(x_0) \neq \emptyset \quad .$$

5. *The **closure** $\overline{E}$ of E is the set of adherent points; it is easy to verify that $E \subseteq \overline{E}$; It is shown that $\overline{E} = E$ if and only if E is closed (exercise 10.b.17).*

6. *A is **dense in** B if $\overline{A} \supseteq B$, that is, if for every $x \in B$ and for every $r > 0$ the intersection $B_r(x) \cap A$ is not empty.*

Note that, having the operational definition (10.b.3) of "open set", then the axioms (in the definition 8.2) in this case become theorems.

Exercises

E10.b.5 `Topics:balls.` [0NZ]

Prove that
$$B_\rho(x) \subseteq B_r(x_0) \qquad (10.b.6)$$
for every $x \in B_\rho(x_0)$ and for every $0 < \rho \leq r - d(x, x_0)$. *Hidden solution:* *[UNACCESSIBLE UUID '0P0']*

E10.b.7 `Topics:balls, disks.` Let $x_1, x_2 \in X$, $r_1, r_2 > 0$, if $d(x_1, x_2) \geq r_1 + r_2$ then [0P1]
$$B_{r_1}(x_1) \cap D_{r_2}(x_2) = \emptyset \quad . \qquad (10.b.8)$$
Hidden solution: *[UNACCESSIBLE UUID '0P2']*

E10.b.9 `Topics:interior. Prerequisites:10.b.5.` Show that $B_r(x)$ is an open set using the definition (10.b.3). *Hidden solution:* *[UNACCESSIBLE UUID '0P4']* [0P3]

E10.b.10 Prove that a metric space is T_2 i.e. Hausdorff (see definition in 8.4). [0P5]

E10.b.11 If $A = B^c$ then show that $(\overline{B})^c = A^\circ$ (using the definitions in this section). [0P6]
Hidden solution: *[UNACCESSIBLE UUID '0P7']*

E10.b.12 `Prerequisites:10.b.11.` Show that the notions of *interior* and *closure* seen above are equivalent to those presented in the definition 8.2. [0P8]

E10.b.13 `Topics:interior.` Show that E is open if and only if $E^\circ = E$. *Hidden solution:* *[UNACCESSIBLE UUID '0PC']* [0PB]

E10.b.14 `Topics:interior.` Show that if $A \subseteq B \subseteq X$ and A is open then $A \subseteq B^\circ$ using the above definitions. [0PD]
Hidden solution: *[UNACCESSIBLE UUID '0PF']*

E10.b.15 `Topics:interior.` Show that if $A \subseteq B \subseteq X$ then $A^\circ \subseteq B^\circ$. *Hidden solution:* [0PG]
[UNACCESSIBLE UUID '0PH']

E10.b.16 `Topics:interior.Prerequisites:10.b.9,10.b.14.` [0PJ]

Given X metric space and $A \subseteq X$, show that

$$A^\circ = (A^\circ)^\circ \quad ,$$

using the above definitions.

For what has been said in 10.b.13, this is equivalent to saying that A° is an open set.

(For the case of X topological space, see the 8.11)

Hidden solution: *[UNACCESSIBLE UUID '0PK']*

E10.b.17 `Topics:interior.` [0PM]

Show that $\overline{E} = E$ if and only if E is closed.

E10.b.18 `Topics:closure. Prerequisites:10.b.15,10.b.11.`*(Replaces 0PN)* [0PP]

Show that if $B \subseteq A \subseteq X$ then $\overline{B} \subseteq \overline{A}$; using the above definitions, or by switching to complement set and using 10.b.15.

E10.b.19 `Topics:closure.Prerequisites:10.b.11, 10.b.18.` [0PQ]

Given a metric space X and a set $A \subseteq X$, show that

$$\overline{A} = \overline{\left(\overline{A}\right)}$$

either by transitioning to the complement set and using 10.b.16, or by using the definition of $\overline{A}$ as *"set of adherent points"*.

As discussed in 10.b.17, this is equivalent to saying that $\overline{A}$ is a closed set.

E10.b.20 Let $E \subseteq X$, then E is a metric space with the restricted distance $\tilde{d} = d|_{E\times E}$. [0PR]

Show that $A \subseteq E$ is open in $(E, \tilde{d})$ (as defined at the beginning of this section) if and only there exists a set $V \subseteq X$ open in (X, d) such that $V \cap E = A$.

(The second way of defining "open" is used in topology.)

Hidden solution: *[UNACCESSIBLE UUID '2GD']*

E10.b.21 `Prerequisites:8.h.5.`Let X be a set with two distances d_1, d_2; let's call τ_1, τ_2 respectively the induced topologies. We have that $\tau_1 \subseteq \tau_2$ if and only if [0PS]

$$\forall x \in X\ \forall r_1 > 0\ \exists r_2 > 0 \ :\ B^2(x, r_2) \subseteq B^1(x, r_1)$$

where

$$B^2(x, r_2) = \{y \in X \ :\ d^2(x, y) < r_2\} \quad , \quad B^1(x, r_1) = \{y \in X \ :\ d^1(x, y) < r_1\} \quad .$$

Note that this exercise is the analogue in metric spaces of the principle 8.h.5 for the bases of topologies.

E10.b.22 Prerequisites:8.h.10, 10.b.21, 12.8, 12.a.4,12.a.5. [0PT]

Having fixed $(X_1, d_1), \ldots, (X_n, d_n)$ metric spaces, let $X = X_1 \times \cdots \times X_n$.

Let φ be one of the norms defined in eqn. (§12.a) in Sec. §12.a. Two possible examples are $\varphi(x) = |x_1| + \cdots + |x_n|$ or $\varphi(x) = \max_{i=1\ldots n} |x_i|$.

Finally, let's define for $x, y \in X$

$$d(x,y) = \varphi\big(d_1(x_1,y_1), \ldots, d_n(x_n,y_n)\big) \quad . \tag{10.b.23}$$

Show that d is a distance; show that the topology in (X, d) coincides with the product topology (see 8.h.10).

Note that this approach generalizes the definition of the Euclidean distance between points in $\mathbb{R}^n$ (taking $X_i = \mathbb{R}$ and $\varphi(z) = \sqrt{\sum_i |z_i|^2}$). We deduce that the topology of $\mathbb{R}^n$ is the product of the topologies of $\mathbb{R}$.

Hidden solution: *[UNACCESSIBLE UUID '0PX']*

See also the exercise 10.b.33, which reformulates the above using the concept of *bases of topologies*.

E10.b.24 Prerequisites:10.b.7,10.a.16. [0PY]

Let $D(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) \le r\}$ be the disk, show that it is closed.

Let $S(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) = r\}$ be the sphere, show that it is closed.

Hidden solution: *[UNACCESSIBLE UUID '0PZ']*

E10.b.25 Prerequisites:10.b.26,10.b.9,10.b.24, 10.b.18.Let $r > 0$. [0Q0]

Let $D(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) \le r\}$ be the disk; show that $\overline{B(x,r)} \subseteq D(x,r)$ and that $B(x,r) \subseteq D(x,r)^{\circ}$.

Let $S(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) = r\}$ be the sphere; show that $\partial B(x,r) \subseteq S(x,r)$.

Find examples of metric spaces in which the above equalities (one, or both) do not hold.

Find an example of a metric space where there is a disk that is open[†77].

(See also 10.g.1 for the case of space $\mathbb{R}^n$). *Hidden solution:* *[UNACCESSIBLE UUID '0Q1'][UNACCESSIBLE UUID '0Q2']*

E10.b.26 Prerequisites:8.a.9.Let $A \subseteq X$ where (X, d) is a metric space, we have that $x \in \partial A$ if and only if there exists $(y_n) \subseteq A$ and $(z_n) \subseteq A^c$ sequences such that $y_n \to x$ and $z_n \to x$. *Hidden solution:* *[UNACCESSIBLE UUID '0Q4']* [0Q3]

E10.b.27 Prerequisites:Section §8.i. Find an example of a metric space (M, d) that does not satisfy the second axiom of countability, *i.e.* such that there is no countable base for the topology associated with (M, d). [0Q5]

Hidden solution: *[UNACCESSIBLE UUID '0Q6']*

E10.b.28 Prerequisites:Section §8.i.Let (M, d) be a metric space and suppose that there exists $D \subseteq M$ that is countable and dense. Such (M, d) is called *separable*. Show that (M, d) satisfies the second axiom of countability. [0Q7]

The converse is true in any topological space, see 8.i.4.

E10.b.29 Prerequisites:10.b.14,10.b.18, 8.13, 10.b.11.Difficulty:*. [0Q8]

Let X be a metric space, and $A \subseteq X$. We want to study the "open-close" operation $\overline{(A^\circ)}$ (which is the closure of the interior of A).

- Show a simple example where $\overline{(A^\circ)}$ is not contained A.
- Then write a characterization of $\overline{(A^\circ)}$ using sequences and balls.
- Use it to show that the "open-close" operation is idempotent, that is, if $D = \overline{(A^\circ)}$ and then $E = \overline{(D^\circ)}$ then $E = D$.

Hidden solution: [UNACCESSIBLE UUID '0Q9'][UNACCESSIBLE UUID '0QB']

E10.b.30 Prerequisites:10.d.3.Show that, for every closed set $C \subseteq X$ there exist countably many open sets A_n such that $\bigcap_n A_n = C$. [0QC]

Hidden solution: [UNACCESSIBLE UUID '0QD']

A set obtained as an intersection of countably many open sets is known as "a G_δ set". The previous exercise shows that in a metric space every closed is a G_δ.

Passing to the complement set, one obtains this statement. A set that is union of countably many closed sets is known as "an F_σ set". The previous exercise shows that in a metric space every open set is an F_σ set.

See also the section §14.d.

E10.b.31 Difficulty:**.Find an example of a metric space where for every $x \in X, r > 0$, $B_r(x)$ is a closed set, but the associated topology is not discrete. [†78] [0QF]

Hidden solution: [UNACCESSIBLE UUID '0QG']

We note that such a space must be *totally disconnected* as shown in 8.e.20.

§10.b.a Bases composed of balls

To face these exercises it is necessary to know the concepts seen in Sec. §8.h.

Exercises

E10.b.32 Prerequisites:8.h.7, 8.h.8.Show that the intersection of two balls is an open set (according to the definition 10.b.2). Hence the family of all balls meets the requirements *(a)* and *(b)* in exercise 8.h.7; so (as shown in 8.h.8), the family of balls is a base for the topology that it generates (which is the topology associated with metric space). [0QJ]

Hidden solution: [UNACCESSIBLE UUID '0QK']

E10.b.33 Let's review the exercise 10.b.22. [0QM]

Having fixed $(X_1, d_1), \ldots, (X_n, d_n)$ metric spaces, let $X = X_1 \times X_1 \times \cdots \times X_n$.

Let d be the distance

$$d(x, y) = \max_{i=1,\ldots n} d_i(x_i, y_i) \quad .$$

[†77] There are also spaces where every ball is closed, see 10.b.31.
[†78] See 8.5 for the definition.

This is the same d defined as in eqn. (10.b.23) inside 10.b.22, setting $\varphi(x) = \max_{i=1\ldots n} |x_i|$. We indicate with $B^d(x,r)$ the ball in (X,d) of center $x \in X$ and radius $r > 0$.

We want to show that d induces the product topology on X, using the results seen in Sec. §8.h.

Taken $t \in X_i, r > 0$ we indicate with $B^{d_i}(t,r)$ the ball in metric space (X_i, d_i). Let $\mathcal{B}_i$ be the family of all balls in (X_i, d_i).

Let $\mathcal{B}$ be defined as

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} B^{d_i}(x_i, r_i) \; : \; \forall i, x_i \in X_i, r_i > 0 \right\}$$

This is the same $\mathcal{B}$ defined in 8.h.11.

Show that every ball $B^d(x,r)$ in (X,d) is the Cartesian product of balls $B^{d_i}(x_i,r)$ in (X_i, d_i). So let $\mathcal{P}$ be the family of balls $B^d(x,r)$ in (X,d).

From 10.b.32 we know that $\mathcal{P}$ is a base for the standard topology in the metric space (X,d).

Use 8.h.5 to show that $\mathcal{P}$ and $\mathcal{B}$ generate the same topology τ.

Use 8.h.11 to prove that τ is the product topology.

We conclude that the distance d generates the product topology.

§10.b.b Accumulation points, limit points

Let's redefine this notion (a special case of what we saw in 8.a.3)

Definition 10.b.34 (accumulation point). *Given $A \subseteq X$, a point $x \in X$ is an* accumulation point *for A if, for every $r > 0$, $B(x,r) \cap A \setminus \{x\}$ is not empty.* [0QN]

The set of accumulation points of A is called **derived set**, we will indicate it with $D(A)$.

Exercises

E10.b.35 `Topics:adherent point, accumulation point.` [0QP]

Check that

- Each accumulation point is also an adherent point, in symbols $D(A) \subseteq \overline{A}$;
- if a point adhering to A is not in A then it is an accumulation point;

So we obtain that $\overline{A} = A \cup D(A)$.

E10.b.36 Given $A \subseteq X$, a point $x \in X$ is an accumulation point if and only if there exists a sequence $(x_n) \subseteq A$ which is injective and such that $\lim_{n\to\infty} x_n = x$. [0QR]

E10.b.37 Let (X,d) metric space, and $x \in X$. Show that $A = \{x\}$ is closed; and that A has an empty inner part if and only if x is accumulation point. *Hidden solution:* *[UNACCESSIBLE UUID '0QT']* [0QS]

E10.b.38 Let $A \subseteq X$ and let $D(A)$ be the *derivative* (i.e. the set of its accumulation points). Show that $D(A)$ is closed. *Hidden solution:* *[UNACCESSIBLE UUID '0QW']* [0QV]

Let's add this definition (a special case of 8.f.2).

Definition 10.b.39 (limit point). *Given a sequence $(x_n)_n \subseteq X$, a point $x \in X$ is said to be a* limit point *for $(x_n)_n$ if there is a subsequence n_k such that $\lim_{k\to\infty} x_{n_k} = x$.* [0QX]

In English literature the terms *"cluster point"*, *"limit point"* and *"accumulation point"* are sometimes considered synonimous, which can be confusing. We will stick to the proposed definitions 10.b.34 and 10.b.39.

Exercises

E10.b.40 Find an example of a metric space (X, d) and a bounded sequence $(x_k)_k \subseteq X$ that has a single limit point x but that does not converge. [0QY]

See also 10.g.2.

E10.b.41 Prerequisites:10.a.5,10.a.12. [0QZ]

- If a sequence $(a_k)_k \subseteq X$ converges to x then it has an unique limit point, which is x.
- If a Cauchy sequence $(a_k)_k \subseteq X$ has a limit point then there is only one limit point x and $\lim_k a_k = x$.

Hidden solution: [UNACCESSIBLE UUID '0R0']

E10.b.42 Topics:perfect set.Prerequisites:10.b.35,3.j.27,8.h.16.Difficulty:**. [2F3]

Suppose (X, d) is a complete metric space. A closed set $E \subseteq X$ without isolated points, *i.e.* consisting only of accumulation points, is called a **perfect set**.

Let C be the Cantor set. Assume that E is perfect and non-empty. Show that there exists a continuous function $\varphi : C \to E$ that is an homeomorphism with its image. This implies that $|E| \geq |\mathbb{R}|$.

So, in a sense, any non-empty perfect set contains a copy of the Cantor set.

This can be proven without relying on continuum hypothesis *3.j.27. Cf. 10.k.6.*

Due to 8.d.5, it is enough to show that there exists a $\varphi : C \to E$ continuous and injective.

Hidden solution: [UNACCESSIBLE UUID '2F4']

Other exercises on these topics are 10.f.5, 10.f.6, 10.f.7, 10.g.2 and 10.g.8.

§10.c Quotients [2C3]

Exercises

E10.c.1 Suppose that d satisfies all distance requirements except *"separation property"*. Consider the relation $\sim$ on X defined as $x \sim y \iff d(x, y) = 0$; show that is an equivalence relation. Let's define $Y = X/\sim$; show that the function d "passes to the quotient", that is, there exists $\tilde{d} : Y \times Y \to [0, \infty)$ such that, for every choice of classes $s, t \in Y$ and every choice of $x \in s, y \in t$ you have $\tilde{d}(s, t) = d(x, y)$. Finally, show that $\tilde{d}$ is a distance on Y. [0R2]

This procedure is the metric space equivalent of *Kolmogoroff quotient*.

E10.c.2 Let (X, d) be a metric space and $\sim$ an equivalence relation on X. Let $Y = X/\sim$ be the quotient space. We define the function $\delta : Y^2 \to \mathbb{R}$ as [0R3]

$$\delta(x, y) = \inf\{d(s, t) : s \in x, t \in y\} . \qquad (10.c.3)$$

Is it a distance on Y? Which properties does it enjoy among those indicated in 10.a.1? *Hidden solution:* *[UNACCESSIBLE UUID '0R4']*

E10.c.4 Let (X, d) be a metric space where X is also a group. Let Θ be a subgroup. [0R5]

We define that $x \sim y \iff xy^{-1} \in \Theta$. It is easy to verify that this is an equivalence relation. Let $Y = X/\sim$ be the quotient space. †79

Suppose d is invariant with respect to left multiplication by elements of Θ:

$$d(x, y) = d(\theta x, \theta y) \ \forall x, y \in X, \forall \theta \in \Theta . \qquad (10.c.5)$$

(This is equivalent to saying that, for every fixed $\theta \in \Theta$ the map $x \mapsto \theta x$ is an isometry). We define the function $\delta : Y^2 \to \mathbb{R}$ as in (10.c.3).

- Show that, taken $s, t \in X$,

$$\delta([s], [t]) = \inf\{d(s, \theta t) : \theta \in \Theta\} \qquad (10.c.6)$$

  where $[s]$ is the class of elements equivalent to s.
- Show that $\delta \geq 0$, that δ is symmetric and that δ satisfies the triangle inequality.
- Suppose that, for every fixed $t \in X$, the map $\theta \mapsto \theta t$ is continuous from Θ to X; suppose also that Θ is closed: then δ is a distance. †80

Hidden solution: *[UNACCESSIBLE UUID '0R6']*

§10.d Distance function [2C4]

Definition 10.d.1. *Given a metric space (M, d), given $A \subset M$ non-empty, we define the* ***distance function*** *$d_A : M \to \mathbb{R}$ as* [0R8]

$$d_A(x) = \inf_{y \in A} d(x, y) . \qquad (10.d.2)$$

Exercises

E10.d.3 `Topics:distance function.` [0R9]

1. Show that d_A is a Lipschitz function.
2. Show that $d_A \equiv d_{\overline{A}}$.
3. Show that $\{x, d_A(x) = 0\} = \overline{A}$.
4. If $M = \mathbb{R}^n$ and A is closed and non-empty, show that the infimum in (10.d.2) is a minimum.

See also 15.d.6 and 15.d.7. *Hidden solution:* *[UNACCESSIBLE UUID '0RB']*

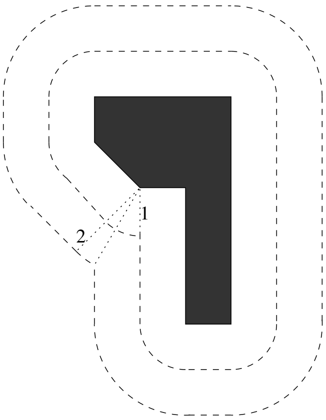


Figure 4: Fattening of a set; exercise 10.d.4

E10.d.4 `Topics:fattened set.Prerequisites:10.d.3.` [ORC]

Consider a metric space (M, d). Let $A \subseteq M$ be closed and non-empty, let $r > 0$ be fixed, and let d_A be the distance function defined as in eqn. (10.d.2). Let then $E = \{x, d_A(x) \le r\}$, notice that it is closed.

1. Show that
$$d_E(x) \ge \max\{0, (d_A(x) - r)\} \ . \tag{10.d.5}$$
2. Show that in (10.d.5) you have equality if $M = \mathbb{R}^N$.
3. Give a simple example of a metric space where equality in (10.d.5) does not hold.
4. If $M = \mathbb{R}^n$, given $A \subset \mathbb{R}^n$ closed non-empty, show that $E = A \oplus D_r$ where $D_r \stackrel{\text{def}}{=} \{x, |x| \le r\}$ and
$$A \oplus B \stackrel{\text{def}}{=} \{x + y, x \in A, y \in B\}$$
is the *Minkowski sum* of the two sets (see also Section §12.f).

Hidden solution: *[UNACCESSIBLE UUID 'ORD']* The set $\{x, d_A(x) \le r\} = A \oplus D_r$ is sometimes called the *"fattening"* of A. In figure 4 we see an example of a set A fattened to $r = 1, 2$; the set A is the black polygon (and is filled in), whereas the dashed lines in the drawing are the contours of the fattened sets. †81 See also the properties in sections §12.f and §12.g.

§10.e Connected set [2C5]

See definitions in Sec. §8.e. We also define this notion.

Definition 10.e.1. *A topological space (X, τ) is* "path connected" *if, for every $x, y \in X$, there is a continuous arc $\gamma : [a, b] \to X$ with $x = \gamma(a), y = \gamma(b)$.* [ORG]

†79 If Θ is a normal subgroup then $Y = X/\sim$ is also written as $Y = X/\Theta$, and this is a group.

†80 Note that, using 14.b.14, under these hypotheses the map of multiplication $(\theta, x) \mapsto \theta x$ is continuous from $\Theta \times X$ to X.

†81 The fattened sets are not drawn filled — otherwise they would cover A.

Exercises

E10.e.2 Find a sequence of connected closed sets $C_n \subseteq \mathbb{R}^2$ such that $C_{n+1} \subseteq C_n$ and the intersection $\bigcap_n C_n$ is a non-empty and disconnected set. [ORH]

Can you find such an example in $\mathbb{R}$?

Hidden solution: [UNACCESSIBLE UUID 'ORJ']

E10.e.3 Find a sequence of sets $C_n \subseteq \mathbb{R}^2$ that are closed and path connected, such that $C_{n+1} \subseteq C_n$ and the intersection $\bigcap_n C_n$ is non-empty, connected, but not path connected. [ORK]

Hidden solution: [UNACCESSIBLE UUID 'ORM'][UNACCESSIBLE UUID 'ORN']

E10.e.4 Consider the example of the set $E \subseteq \mathbb{R}^2$ given by [ORP]

$$E = \left\{(0,t) \,:\, -1 \le t \le 1\right\} \cup \left\{\left(x, \sin\frac{1}{x}\right) \,:\, x \in (0,1]\right\} \quad . \tag{10.e.5}$$

Show that this set is closed, connected, but is not path connected.

Hidden solution: [UNACCESSIBLE UUID 'ORQ']

This set is sometimes called *closed topologist's sine curve* [45].

E10.e.6 Difficulty:*.Let (X, d) be a metric space. Show that $E \subseteq X$ is disconnected if and only if *"there are two disjoint open sets, each of which intersect E and such that E is covered by their union"* (see the proposition formalized in eqn. (8.e.12) in the exercise 8.e.11). [ORR]

Hidden solution: [UNACCESSIBLE UUID 'ORS']

E10.e.7 Let $D \subseteq \mathbb{R}^2$ be countable; show that $\mathbb{R}^2 \setminus D$ is path connected. [ORT]

Hidden solution: [UNACCESSIBLE UUID 'ORV']

E10.e.8 Find an example of a metric space X that is path connected, where there exists an open subset $A \subseteq X$ that is connected but not path connected. *Hidden solution:* [UNACCESSIBLE UUID 'ORZ'] [ORY]

§10.f Topology in the real line [2C6]

Exercises

E10.f.1 Show that a set $A \subseteq \mathbb{R}$ is an interval if and only it is convex, if and only if it is connected. [OSO]

(A part of the proof is in Theorem 5.11.3 in [2]).

Hidden solution: [UNACCESSIBLE UUID 'OS1']

(Note how in this case the exercises 3.d.46 and 8.e.15 coincide).

E10.f.2 Let us fix $\alpha \in \mathbb{R}$, consider the set A of numbers of the form $\alpha n + m$ with n, m integers. Show that A is dense in $\mathbb{R}$ if and only if α is irrational. *Hidden solution:* [UNACCESSIBLE UUID 'OS3'] [OS2]

E10.f.3 Given $I \subseteq \mathbb{Q}$ non-empty, show that I is connected if and only I contains only one point. *Hidden solution:* [UNACCESSIBLE UUID 'OS5'] [OS4]

E10.f.4 Show that every open non-empty set $A \subset \mathbb{R}$ is the union of a family (at most countable) of disjoint open intervals. *Hidden solution:* *[UNACCESSIBLE UUID '0S7']* [0S6]

E10.f.5 Find a compact $A \subset \mathbb{R}$ that has a countable number of accumulation points. *Hidden solution:* *[UNACCESSIBLE UUID '0S9']* [0S8]

E10.f.6 Prerequisites:10.b.36. Show that the set $A \subset \mathbb{R}$ defined by [0SB]

$$A = \{0\} \cup \{1/n : n \in \mathbb{N}, n \geq 1\} \cup \{1/n + 1/m : n, m \in \mathbb{N}, n \geq 1, m \geq 1\}$$

is compact; identify its accumulation points.

Hidden solution: *[UNACCESSIBLE UUID '0SC']*

E10.f.7 Difficulty:**. Let $A \subset \mathbb{R}$. We recall that $D(A)$ is the derivative of A (i.e. the set of accumulation points of A). Describe a closed set A such that the sets [0SD]

$$A, D(A), D(D(A)), D(D(D(A))) \ldots$$

are all different.

Hidden solution: *[UNACCESSIBLE UUID '0SF']*

E10.f.8 Prerequisites:10.b.16, 10.b.19, 8.13, 10.b.29.Difficulty:**. [0SG]

Find a subset A of $\mathbb{R}$ such that the following 7 subsets of $\mathbb{R}$ are all distinct:

$$A,\ \overline{A},\ A^\circ,\ \left(\overline{A}\right)^\circ,\ \overline{(A^\circ)},\ \overline{\left(\left(\overline{A}\right)^\circ\right)},\ \left(\overline{(A^\circ)}\right)^\circ\ .$$

Also prove that no other different sets can be created by continuing in the same way (also replacing $\mathbb{R}$ with a generic metric space).

Hidden solution: *[UNACCESSIBLE UUID '0SH']*

E10.f.9 Difficulty:**.Prove that it is not possible to write $\mathbb{R}$, or an interval $D \subseteq \mathbb{R}$, as a countable and infinite union of closed and bounded intervals, pairwise disjoint. [0W6]

Hidden solution: *[UNACCESSIBLE UUID '0W7']*

§10.g Topology in Euclidean spaces [2C7]

In the following we consider the metric space $\mathbb{R}^n$ with the usual Euclidean distance.

Exercises

E10.g.1 Prerequisites:10.b.25.Let $B(x,r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : |x-y| < r\}$ be the ball; let $D(x,r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : |x-y| \leq r\}$ the disc; let $S(x,r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : |x-y| = r\}$ be the sphere. Show that $\overline{B(x,r)} = D(x,r)$, that $B(x,r) = D(x,r)^\circ$, and that $\partial B(x,r) = S(x,r)$. Also show that $B(x,r)$ is not closed and $D(x,r)$ is not open. [0SM]

(This result holds more generally in any normed space: see 12.7).

E10.g.2 Prerequisites:10.b.41, 10.a.9. Given a sequence $(x_k)_k \subseteq \mathbb{R}^n$, these facts are equivalent [0SN]

a the sequence is bounded and has a single limit point x

b $\lim_k x_k = x$.

Hidden solution: [UNACCESSIBLE UUID '0SP'] See also 10.b.40.

E10.g.3 Prerequisites:8.a.9, 8.a.10.For each $A \subseteq \mathbb{R}^n$ closed non-empty set, there exists $B \subseteq A$ such that $A = \partial B$. [0SQ]

In which cases does there exist such a B that is countable?

In which cases does there exist such a B that is closed?

Hidden solution: [UNACCESSIBLE UUID '0SR'][UNACCESSIBLE UUID '0SS']

See also 8.i.4.

E10.g.4 Prerequisites:8.a.13.For every non-empty closed set $E \subseteq \mathbb{R}^N$, there exists $F \subseteq \mathbb{R}^n$ such that $E = D(F)$. [0SV]

Can you find it $F \subseteq E$?

Hidden solution: [UNACCESSIBLE UUID '0SX'][UNACCESSIBLE UUID '0SY']

E10.g.5 What are the sets $A \subset \mathbb{R}^n$ that are both open and closed? [0T0]

Hidden solution: [UNACCESSIBLE UUID '0T2']

E10.g.6 Let $f : \mathbb{R} \to \mathbb{R}^n$ continue; show that these two conditions are equivalent [0T3]

- $\lim_{t\to\infty} |f(t)| = +\infty$ and $\lim_{t\to-\infty} |f(t)| = +\infty$;
- f is **proper**, i.e. for every compact $K \subset \mathbb{R}^n$ we have that the counterimage $f^{-1}(K)$ is a compact of $\mathbb{R}$.

E10.g.7 Prerequisites:Section §8.i.Show that $\mathbb{R}^N$ satisfies the second axiom of countability. [0T4]

E10.g.8 Prerequisites:8.i.3. Note:*exercise 4 in the written exam of 13/1/2011*. [0T5]

If $A \subseteq \mathbb{R}^n$ is composed only of isolated points, then A has countable cardinality.

Conversely, therefore, if $A \subseteq \mathbb{R}^n$ is uncountable then the derivative $D(A)$ is not empty.

Hidden solution: [UNACCESSIBLE UUID '0T6']

E10.g.9 Let $A \subset \mathbb{R}^n$ be a bounded set. For every $\varepsilon > 0$ there is a set $I \subset A$ that satisfies: [0T7]

- I is a finite set,
- $\forall x, y \in I$, $x \neq y$ you have $x \notin B(y, \varepsilon)$ (i.e. $d(x, y) \geq \varepsilon$),
- $$A \subseteq \bigcup_{x\in I} B(x, \varepsilon) \ .$$

Hidden solution: [UNACCESSIBLE UUID '0T8']

E10.g.10 Difficulty:*.What is the cardinality of the family of open sets in $\mathbb{R}^n$? [0T9]

Hidden solution: [UNACCESSIBLE UUID '0TB']

E10.g.11 Let $E \subseteq \mathbb{R}^n$ be not empty and such that every continuous function $f : E \to \mathbb{R}$ admits maximum: show that E is compact. [0TD]

(See 10.j.10 for generalization to metric spaces)

Hidden solution: [UNACCESSIBLE UUID '0TF']

§10.h Fixed points [2C8]

Exercises

E10.h.1 Find a function $f : \mathbb{R} \to \mathbb{R}$ such that [0TG]

$$\forall x, y \in \mathbb{R}, x \neq y \Rightarrow |f(x) - f(y)| < |x - y|$$

but that has no fixed points. *Hidden solution:* *[UNACCESSIBLE UUID '27D']*

E10.h.2 `Prerequisites:10.j.11`. Let (X, d) be a compact metric space, and let $f : X \to X$ be such that [0TH]

$$\forall x, y \in X, x \neq y \Rightarrow d(f(x), f(y)) < d(x, y) \quad .$$

Show that f has a single fixed point.

This result is sometimes called *Edelstein's Theorem*.

Hidden solution: *[UNACCESSIBLE UUID '27C']*

§10.i Isometries [2C9]

Definition 10.i.1. *Given (M_1, d_1) and (M_2, d_2) metric spaces, a map $\varphi : M_1 \to M_2$ is called an **isometry** if* [0TK]

$$\forall x, y \in M_1, \ \ d_1(x, y) = d_2(\varphi(x), \varphi(y)) \ . \tag{10.i.2}$$

We will see in Sec. §12.b the same definition in the case of normed vector spaces. Obviously an isometry is Lipschitz, and therefore continuous. Isometries enjoy some properties.

Exercises

E10.i.3 `Topics:isometry`. An isometry is always injective. [0TM]

E10.i.4 If the isometry φ is surjective (and therefore is bijective) then the inverse φ^{-1} is also an isometry. [0TP]

E10.i.5 If (M_1, d_1) is complete then its image $\varphi(M_1)$ is a complete set in M_2; and therefore it is a closed in M_2. [0TQ]

Hidden solution: *[UNACCESSIBLE UUID '0TR']* Consequently, if the isometry φ is bijective and one of the two spaces is complete then the other is also complete.

E10.i.6 `Topics:isometry. Difficulty:*`. Let (X, d) be a compact metric space; let $T : X \to X$ be an isometry, then T is surjective. [0TT]

Provide a simple example of a non-compact metric space and $T : X \to X$ a non-surjective isometry.

Hidden solution: *[UNACCESSIBLE UUID '0TV']*

E10.i.7 `Topics:isometry.Prerequisites:10.i.6.Difficulty:*`. [0TW]

Let (X, d) and (Y, δ) be two metric spaces of which X compact, $T : X \to Y$ and $S : Y \to X$ two *isometries*. Prove that T and S are bijective.

Hidden solution: *[UNACCESSIBLE UUID '0TY']*

E10.i.8 `Topics:isometry.Difficulty:*.` Find an example of two metric spaces (X, d) and (Y, δ) that are not isometric but for which there are two isometries $T : X \to Y$ and $S : Y \to X$. [0TZ]

Hidden solution: *[UNACCESSIBLE UUID '0V1']*

§10.j Compactness [2CB]

The Heine-Borel Theorem [56] extends to this context.

Theorem 10.j.1. *Given a metric space (X, d) and its subset $C \subseteq X$, The following three conditions are equivalent.* [0V3]

- *C is* sequentially compact*: every sequence $(x_n) \subset C$ has a subsequence converging to an element of C.*
- *C is* compact*: from each family of open sets whose union covers C, we can choose a finite subfamily whose union covers C.*
- *C is complete, and is* totally bounded*: for every $\varepsilon > 0$ there are finite points $x_1 ... x_n \in C$ such that $C \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.*

(This theorem has a generalization in topological spaces, see 8.f.7).

Exercises

E10.j.2 Setting $X = \mathbb{R}^n$ and d the usual Euclidean distance, taken $C \subseteq \mathbb{R}^n$, use the above theorem 10.j.1 to show (as a corollary) the usual Heine-Borel theorem [56]: *C is compact if and only if it is closed and bounded.* [0V4]

Hidden solution: *[UNACCESSIBLE UUID '0V5']*

E10.j.3 Show that if $K \subset X$ is compact then it is closed. *Hidden solution:* *[UNACCESSIBLE UUID '0V7']* (See 8.d.3 for the case of topological space) [0V6]

E10.j.4 Let (X, d_X) and (Y, d_Y) be metric spaces, with (X, d_X) compact; suppose that $f : X \to Y$ is continuous and injective. Show that f is a homeomorphism between X and its image $f(X)$. [0V8]

Hidden solution: *[UNACCESSIBLE UUID '0V9']*

(See 8.d.5 for the case of topological space).

E10.j.5 Let $n \geq 1$ be natural. Let (X_i, d_i) be compact metric spaces, for $i = 1, \ldots n$; choose $y_{i,k} \in X_i$ for $i = 1, \ldots n$ and $k \in \mathbb{N}$. Show that there exists a subsequence k_h such that, for every fixed i, y_{i,k_h} converges, that is, the limit $\lim_{h\to\infty} y_{i,k_h}$ exists. [0VB]

E10.j.6 `Difficulty:**.`Let (X_i, d_i) be compact metric spaces, for $i \in \mathbb{N}$, and choose $y_{i,k} \in X_i$ for $i, k \in \mathbb{N}$. Show that there exists a subsequence k_h such that, for every fixed i, y_{i,k_h} converges, that is, the limit $\lim_{h\to\infty} y_{i,k_h}$ exists. [0VC]

E10.j.7 Let be given a metric space (X, d). As in 10.b.1 we define the *disk* $D(x, \varepsilon) \stackrel{\text{def}}{=} \{y \in X, d(x, y) \leq \varepsilon\}$ (which is closed). (X, d) is *locally compact* if for every $x \in X$ there exists $\varepsilon > 0$ such that $D(x, \varepsilon)$ is compact. Consider this proposition. [0VD]

*«**Proposition** A locally compact metric space is complete. **Proof** Let $(x_n)_n \subset X$ be a Cauchy sequence, then eventually its terms are distant at most ε, so they are contained in a small compact disk, so there is a subsequence that converges, and then, by the result 10.a.12, the whole sequence converges. q.e.d. »*

If you think the proposition is true, rewrite the proof rigorously. If you think it's false, find a counterexample.

Hidden solution: [UNACCESSIBLE UUID '0VF']

E10.j.8 Let (X, d) be a metric space, and let $C \subset X$. Show that C is totally bounded if and only if $\overline{C}$ is totally bounded. (See 10.j.1 for the definition of *totally bounded*). *Hidden solution:* [UNACCESSIBLE UUID '0VH'] [0VG]

E10.j.9 Prerequisites:10.b.20.Let (X, d) be a totally bounded metric space. Let $E \subseteq X$, then E is a metric space with the restricted distance $\tilde{d} = d|_{E\times E}$. Show that $(E, \tilde{d})$ is totally bounded. (See 10.j.1 for the definition of *totally bounded*). *Hidden solution:* [UNACCESSIBLE UUID '2GC'] [2GB]

E10.j.10 Prerequisites:10.b.41,10.j.13.Difficulty:*. [0VJ]

Let (X, d) be a metric space such that every continuous function $f : X \to \mathbb{R}$ has maximum: show that the space is compact.

(See 10.g.11 for a rewording with $X = \mathbb{R}^n$.) *Hidden solution:* [UNACCESSIBLE UUID '0VM'][UNACCESSIBLE UUID '0VN']

E10.j.11 Topics:compact.Prerequisites:10.j.3. [0VP]

Let (X, d) be a metric space, and let $A_n \subseteq X$ be compact non-empty subsets such that $A_{n+1} \subseteq A_n$: then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

(This result can be derived from 8.d.4; but try to give a direct demonstration, using the characterization of "compact" as "sequentially compact ", i.e. the first point in 10.j.1).

Hidden solution: [UNACCESSIBLE UUID '0VQ']

E10.j.12 Let be given a metric space (X, d) and its subset $C \subseteq X$ that is *totally bounded*, as defined in 10.j.1: show that C is bounded, i.e. for every $x_0 \in C$ we have [0VR]

$$\sup_{x\in C} d(x_0, x) < \infty \quad ,$$

or equivalently, for every $x_0 \in C$ there exists $r > 0$ such that $C \subseteq B(x_0, r)$.

The opposite implication does not hold, as shown in 10.j.14

E10.j.13 Let (X, d) be a metric space and let $D \subseteq X$, show that these clauses are equivalent: [0VS]

- D is not totally bounded;
- there exists $\varepsilon > 0$ and there is a sequence $(x_n)_n \subseteq D$ for which

$$\forall n, m \in \mathbb{N},\ d(x_n, x_m) \geq \varepsilon \quad .$$

E10.j.14 Prerequisites:10.j.13.Let $X = C^0([0,1])$ be the space of continuous and bounded functions $f : [0,1] \to \mathbb{R}$, endowed with the usual distance [0VT]

$$d_\infty(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x)-g(x)| \quad .$$

We know that (X, d_∞) is a complete metric space. Let

$$D(0,1) = \{f \in X : d_\infty(0,f) \le 1\} = \{f \in X : \forall x \in [0,1], \quad |f(x)| \le 1\}$$

the disk of center 0 (the function identically zero) and radius 1. We know from 10.b.24 that it is closed, and therefore it is complete. Show that D is not totally bounded by finding a sequence $(f_n) \subseteq D$ as explained in 10.j.13.

§10.k Baire's Theorem and categories

The following is *Baire's category theorem*; there are several equivalent statements.

Theorem 10.k.1. *Suppose (X,d) is complete.* [0VV]

- *Given countably many sets A_n that are open and dense in X, we have that $\bigcap_n A_n$ is dense.*
- *Given countably many sets C_n closed with empty interior in X, we have that $\bigcup_n C_n$ has empty interior.*

Definition 10.k.2. *A set that is contained in the union of countably many closed sets with empty interior is called* first category set in X. [†82] *A set that is not* first category, *is said* second category. [0VW]

Exercises

E10.k.3 A complete metric space X is second category in itself. *Hidden solution:* [0VX]
[UNACCESSIBLE UUID '0VY']

E10.k.4 Given $X = \mathbb{R}$, the set of irrational numbers is second category in $\mathbb{R}$. *Hidden solution:* *[UNACCESSIBLE UUID '0W0']* [0VZ]

E10.k.5 Reflect on the statements: [0W1]

- A closed set C inside a complete metric space (X,d) is complete (when viewed as a metric space (C,d)).
- The set $C = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is closed in $\mathbb{R}$, so C is complete with distance $d(x,y) = |x-y|$.
- C is composed of countably many points.
- A singleton $\{x\}$ is a closed set with an empty internal part.

Why is there no contradiction?

Hidden solution: *[UNACCESSIBLE UUID '0W2']*

E10.k.6 Topics:perfect set.Prerequisites:10.b.35,3.j.27. [0W3]

Suppose (X, d) is a complete metric space. A closed set without isolated points, *i.e.* consisting only of accumulation points, is called a **perfect set**. Show that a non-empty perfect set E contained in X must be uncountably infinite. *(Find a simple direct proof, using Baire's Theorem 10.k.1.)*

Hidden solution: *[UNACCESSIBLE UUID '2DZ']*

The Cantor set is a perfect set, see 9.b.1.

§10.l Infinite product of metric spaces

Exercises

E10.l.1 Prerequisites:10.a.8.Sia $\varphi(t) = t/(1+t)$. Let (X_i, d_i) be metric spaces with $i \in \mathbb{N}$, let $X = \prod_{i\in\mathbb{N}} X_i$, for any $f, g \in X$ we define the distance on X as [0W9]

$$d(f, g) = \sum_{k=0}^{\infty} 2^{-k}\varphi(d_i(f(k), g(k)))\ .$$

Prove that d is a distance.

E10.l.2 Let $f, f_n \in X$ be as before in 10.l.1, show that $f_n \to_n f$ according to this metric if and only if for every k we have $f_n(k) \to_n f(k)$. [0WB]

E10.l.3 Let (X_i, d_i) and (X, d) be as before in 10.l.1. If all the spaces (X_i, d_i) are complete, prove that (X, d) is complete. [0WC]

E10.l.4 Prerequisites:10.j.6,10.l.2.Difficulty:*.Let (X_i, d_i) and (X, d) be as before in 10.l.1. If all the spaces (X_i, d_i) are compact, prove that (X, d) is compact. *Hidden solution:* *[UNACCESSIBLE UUID '0WF']* [0WD]

E10.l.5 Prerequisites:10.l.4.We want to define a distance for the space of sequences. We proceed as in 10.l.1. We choose $X_i = \mathbb{R}$ for each i and set that d_i is the Euclidean distance; then for $f, g : \mathbb{N} \to \mathbb{R}$ we define [0WG]

$$d(f, g) = \sum_k 2^{-k}\varphi(|f(k) - g(k)|)\ .$$

We have constructed a metric space of sequences $(\mathbb{R}^{\mathbb{N}}, d)$.

In the space of sequences $(\mathbb{R}^{\mathbb{N}}, d)$ we define

$$K = \{f \in \mathbb{R}^{\mathbb{N}}, \forall k, |f(k)| \le 1\} \quad .$$

Show that K is compact. *Hidden solution:* *[UNACCESSIBLE UUID '0WH']*

E10.l.6 Let $N(\rho)$ be the minimum number of radius balls ρ that are needed to cover K (from the previous exercise 10.l.5). Estimate $N(\rho)$ for $\rho \to 0$. [0WJ]

See also Sec. §11

§10.m Ultrametric

Definition 10.m.1. *An* ultrametric space *is a metric space in which the triangle inequality is reinforced by the condition* [0WM]

$$d(x,y) \le \max\{d(x,z), d(z,y)\} \ . \qquad (10.m.2)$$

Exercises

E10.m.3 Show that (10.m.2) implies that d satisfies the triangle inequality. [0WN]

E10.m.4 Note that if $d(x,y) \neq d(y,z)$ then $d(x,z) = \max\{d(x,y), d(y,z)\}$. *Hidden solution: [UNACCESSIBLE UUID '0WQ']* Intuitively, all triangles are isosceles, and the base is shorter than equal sides. [0WP]

E10.m.5 Consider two balls $B(x,r)$ and $B(y,\rho)$ radius $0 < r \le \rho$ that have non-empty intersection: then $B(x,r) \subseteq B(y,\rho)$. [0WR]

Similarly for the disks $D(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) \le r\}$ and $D(y,r)$.

Hidden solution: [UNACCESSIBLE UUID '0WS']

E10.m.6 Show that two balls $B(x,r)$ and $B(y,r)$ of equal radius are disjoint or are coincident; in particular they are coincident if and only if $y \in B(x,r)$. Similarly for the discs $D(x,r) \stackrel{\text{def}}{=} \{y \in X : d(x,y) \le r\}$ and $D(y,r)$. [0WT]

Hidden solution: [UNACCESSIBLE UUID '0WV']

E10.m.7 Show that every open ball $B(x,r)$ is also closed. Show that every disk $D(x,r)$ with $r > 0$ is also open. *Hidden solution: [UNACCESSIBLE UUID '0WX']* By the exercise 8.e.20, there follows that the space is *totally disconnected*. [0WW]

E10.m.8 Let $\varphi : [0,\infty) \to [0,\infty)$ be a function that is continuous in zero, monotonically weakly increasing and with $\varphi(x) = 0 \iff x = 0$. Show that $\tilde d = \varphi \circ d$ is still an ultrametric. Show that spaces (X,d) $(X,\tilde d)$ have the same topology. [0WY]

Compare with the exercise 10.a.8, notice that we do not require φ to be subadditive.

§10.m.a Ultrametric space of sequences

Let's build this example of *ultrametric* on the space of sequences.

Definition 10.m.9. *Let I be a non-empty set, with at least two elements. Let $X = \{f : \mathbb{N} \to I\} = I^{\mathbb{N}}$ be the space of sequences. Let $x,y \in X$. If $x = y$ then we set $d(x,y) = 0$.* [†83] *If $x \neq y$, we set* [0X0]

$$c(x,y) = \min\{n \ge 0, x(n) \neq y(n)\} \qquad (10.m.10)$$

to be the first index where the sequences are different; then we define $d(x,y) = 2^{-c(x,y)}$.

Remark 10.m.11. *Because of the exercise 10.m.8, we could equivalently define $d(x,y) = \varepsilon_{c(x,y)}$ with $\varepsilon_n > 0$ infinitesimal decreasing sequence.* [0X1]

Exercises

E10.m.12 Prerequisites:10.m.9.Show that $d(x, y) \leq \max\{d(x, z), d(y, z)\}$. [0X2]

Hidden solution: *[UNACCESSIBLE UUID '0X3']*

E10.m.13 Topics:complete. Prerequisites:10.m.9.Show that (X, d) is complete. *Hidden solution:* *[UNACCESSIBLE UUID '0X5']* [0X4]

E10.m.14 Topics:compact. [0X6]

Prerequisites:10.m.9.

Show that (X, d) is compact if and only if I is a finite set.

Hidden solution: *[UNACCESSIBLE UUID '0X7']*

E10.m.15 Prerequisites:10.m.9,9.b.2. Suppose that I is a group; then X is a group (it is the Cartesian product of groups); and multiplication is carried out "component by component". Show that the product in X is a continuous operation, and so for the inversion map. So (X, d) is a *topological group*. [0X8]

Hidden solution: *[UNACCESSIBLE UUID '0X9']*

E10.m.16 Prerequisites:10.m.9,9.b.2. Let I be a set of cardinality 2, then the space (X, d) is homeomorphic to the Cantor set (with the usual Euclidean metric $|x - y|$). [0XC]

Hidden solution: *[UNACCESSIBLE UUID '0XD']*

Combining this result with 10.m.15 we get that the Cantor set (with its usual topology) can be endowed with an abelian group structure, where the sum and inverse are continuous functions; This makes it a topological group.

See also 11.24.

§10.n P-adic ultrametric [2CG]

We report from the notes [2] the definition of the *p–adic distance* on the $\mathbb{Q}$ set. Let p be a fixed prime number.

Definition 10.n.1. *Each rational number $x \neq 0$ breaks down uniquely as a product* [0XF]

$$x = \pm p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} , \tag{10.n.2}$$

*where $p_1 < p_2 < \cdots < p_k$ are prime numbers and the m_j integers. Fixed as above a prime number p, we define the **p–adic absolute value** of $x \in \mathbb{Q}$ as*

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-m} & \text{if } p^m \text{ is the factor with base } p \text{ in the decomposition (10.n.2)} . \end{cases}$$

*Finally, we define $d(x, y) = |x - y|_p$, which will turn out to be a distance on $\mathbb{Q}$, called **p–adic distance**.*

We add this definition, which will be very useful in the following.

Definition 10.n.3. *For $n \in \mathbb{Z}, n \neq 0$ we define* [0XG]

$$\varphi_p(n) = \max\{h \in \mathbb{N}, p^h \text{ divides } n\} .$$

Let's also define $\varphi_p(0) = \infty$. This φ_p is known as p-adic valuation [63]. .

†82 It is sometimes also called *meagre set* (for example in Wikipedia [47]).
†83 This can also be achieved by defining $c(x, x) = \infty$

Exercises

E10.n.4 Prove these fundamental relation. [0XH]

1. $|1|_p = 1$ and more generally $|n|_p \le 1$ for every nonnull integer n, with equality if n is not divisible by p.
2. Given n nonnull integer, we have that $|n|_p = p^{-\varphi_p(n)}$.
3. Given n, m integers, we have that $\varphi_p(n+m) \ge \min\{\varphi_p(n), \varphi_p(m)\}$ with equality if $\varphi_p(n) \ne \varphi_p(m)$.
4. Given n, m nonzero integers, we have that $\varphi_p(nm) = \varphi_p(n)+\varphi_p(m)$ and therefore $|nm|_p = |n|_p|m|_p$.
5. Given $x = a/b$ with a, b nonnull integers we have that $|x|_p = p^{-\varphi_p(a)+\varphi_p(b)}$. Note that if a, b are coprime, then one of the two terms $\varphi_p(a), \varphi_p(b)$ is zero.
6. Prove that $|xy|_p = |x|_p|y|_p$ for $x, y \in \mathbb{Q}$.
7. Prove that $|x/y|_p = |x|_p/|y|_p$ for $x, y \in \mathbb{Q}$ nonzero.

E10.n.5 Check that [0XM]

$$|x + y|_p \le \max\left\{|x|_p, |y|_p\right\} \tag{10.n.6}$$

for each $x,\ y \in \mathbb{Q}$. and therefore

$$d_p(x, z) \le \max\left\{d_p(x, y), d_p(y, z)\right\}, \qquad \forall\, x,\ y,\ z \in \mathbb{Q}\ .$$

that is, this is an ultrametric (and therefore a distance). *Hidden solution:* *[UNACCESSIBLE UUID '0XN']* The properties 6 and (10.n.6) say that the *p-adic valuation* is an absolute value, and indeed it is a *Krull valuation*.

E10.n.7 Show that the multiplication map is continuous. *Hidden solution:* *[UNACCESSIBLE UUID '0XR']* [0XQ]

E10.n.8 Find an example of a sequence that tends to zero (but never takes the value 0). This example shows that the associated topology is not the discrete topology. *Hidden solution:* *[UNACCESSIBLE UUID '0XV']* [0XT]

E10.n.9 `Difficulty:*`.Show, for every $a/b \in \mathbb{Q}$ with a, b coprime and b not divisible by p, there exists $(x_n)_n \subseteq \mathbb{Z}$ such that $|x_n - a/b|_p \to_n 0$. Note that the assumption is necessary. [0XW]

Hidden solution: *[UNACCESSIBLE UUID '0XX']* We proved that $\mathbb{Z}$ is dense in the disk $\{x \in \mathbb{Q}, |x|_p \le 1\}$.

E10.n.10 `Difficulty:**`.Show that $(\mathbb{Q}, d)$ is not a complete metric space. [0XY]

Hidden solution: *[UNACCESSIBLE UUID '0XZ']*

E10.n.11 Show that no p–adic distance on $\mathbb{Q}$ is bi–Lipschitz equivalent to the Euclidean distance (induced by $\mathbb{R}$). [0Y0]

Hidden solution: *[UNACCESSIBLE UUID '0Y1']*

§10.o Circle [2CF]

Definition 10.o.1. *$S^1 = \{x \in \mathbb{R}^2, |x| = 1\}$ is the **circle** in the plane.* [0Y3]

It is a closed set in $\mathbb{R}^2$, so we can think of it as a complete metric space with the Euclidean distance $d(x, y) = |x - y|_{\mathbb{R}^2}$.

Definition 10.o.2. *We denote by $\mathbb{R}/2\pi$ the quotient space $\mathbb{R}/\sim$ where $x \sim y \iff (x-y)/(2\pi) \in \mathbb{Z}$ is an equivalence relation that makes points equivalent that are an integer multiple of 2π. This space $\mathbb{R}/2\pi$ is called **the space of real numbers modulo 2π.*** [0Y4]

As usual, given $t \in \mathbb{R}$, we indicate with $[t]$ the class of elements in $\mathbb{R}/2\pi$ equivalent to t.

Exercises

E10.o.3 Consider the map [0Y5]

$$\begin{aligned} \Phi : \mathbb{R}/2\pi &\to S^1 \\ [t] &\mapsto (\cos(t), \sin(t)) \end{aligned}$$

Show that it is well-defined and bijective.

Hidden solution: *[UNACCESSIBLE UUID '0Y6']*

E10.o.4 Through this bijection we transport the Euclidean distance from S^1 to $\mathbb{R}/2\pi$ defining [0Y7]

$$d_e([s], [t]) = |\Phi([s]) - \Phi([t])|_{\mathbb{R}^2} \ .$$

With this choice the map Φ turns out to be an isometry between (S^1, d) and $(\mathbb{R}/2\pi, d_e)$ (see the Definition 10.i.1). So the latter is a complete metric space.

With some simple calculations it can be deduced that

$$d_e([s], [t]) = \sqrt{|\cos(t) - \cos(s)|^2 + |\sin(t) - \sin(s)|^2} = \sqrt{2 - 2\cos(t - s)} \ .$$

Then we define the function

$$d_a([s], [t]) = \inf\{|s - t - 2\pi k| : k \in \mathbb{Z}\} \ ,$$

show that it is a distance on $\mathbb{R}/2\pi$.

Hidden solution: *[UNACCESSIBLE UUID '0Y8']*

E10.o.5 Show that $d_a([s], [t])$ is the length of the shortest arc in S^1 that connects $\Phi([s])$ to $\Phi([t])$. [0Y9]

E10.o.6 Show that distances d_a and d_e are equivalent, proving that $\frac{2}{\pi} d_a \le d_e \le d_a$. [0YB]

Hidden solution: *[UNACCESSIBLE UUID '0YC']*

E10.o.7 `Prerequisites:`10.c.1.One can easily show that a function $f : \mathbb{R}/2\pi \to X$ can be seen as a periodic function $\tilde{f} : \mathbb{R} \to X$ of period 2π, and vice versa. [0YD]

This can be easily obtained from the relation $f([t]) = \tilde{f}(t)$ where t is a generic element of its equivalence class $[t]$. Assuming that $\tilde{f}$ is periodic (with period 2π), the above relation allows to derive f from $\tilde{f}$ and vice versa.

Show that f is continuous if and only if $\tilde{f}$ is continuous.

E10.o.8 Prerequisites:8.b.3.Let (X, τ) be the *compactified line*, the topological space defined in 8.b.3. Show that it is homeomorphic to S^1. [0YF]

§11 Dimension [0YH]

Let (X, d) be a metric space. Let in the following K a compact non-empty subset of X, and $N(\rho)$ the minimum number of balls of radius ρ that are needed to cover K. [†84]

Definition 11.1. *If the limit exists* [0YJ]

$$\lim_{\rho \to 0+} \frac{\log N(\rho)}{\log(1/\rho)} \tag{11.2}$$

we will say that this limit is the ***Minkowski dimension*** $\dim(K)$ *of* K.

If the limit does not exist, we can still use the limsup and the liminf to define the *upper and lower dimension*.

Note that this definition depends *a priori* on the choice of the distance, i.e. $N = N(\rho, K, d)$ and $\dim = \dim(K, d)$. See in particular 11.10.

Exercises

E11.3 Show that $N(\rho)$ is decreasing as a function of ρ. [0YK]

E11.4 `Prerequisites: 10.j.5.Difficulty:*.`Show that $N(\rho)$ is bounded if and only if K contains only a finite number of points. *Hidden solution:* *[UNACCESSIBLE UUID '0YP']* So if K is infinite, then $\lim_{\rho \to 0+} N(\rho) = \infty$. [0YN]

E11.5 Let $N'(\rho)$ be the minimum number of balls, with radius ρ and centered in K, that are necessary to cover K: then [0YQ]

$$N'(2\rho) \le N(\rho) \le N'(\rho) .$$

So the dimension does not change if you require the balls to be centered at points of K. *Hidden solution:* *[UNACCESSIBLE UUID '0YR']*

E11.6 Let $P(\rho)$ be the maximum number of balls, with radius ρ and centered in K, that are disjoint. Show that [0YS]

$$N(2\rho) \le P(\rho) \le N(\rho/2) .$$

So the dimension can also be calculated as

$$\lim_{\rho \to 0+} \frac{\log P(\rho)}{\log(1/\rho)} . \tag{11.7}$$

Such a construction is known as *ball packing*. *Hidden solution:* *[UNACCESSIBLE UUID '0YT']*

E11.8 In the exercise 11.6 it is important to require that the balls are centered in points of K. Find an example of metric space (X, d) and compact $K \subseteq X$ of finite dimension, but such that, for every $n \in \mathbb{N}$ and every $\rho > 0$, there are infinite disjoint balls each containing only one point of K. [0YV]

Hidden solution: *[UNACCESSIBLE UUID '0YW']*

E11.9 Show that the dimension does not change if you use disks [0YX]

$$D(x, r) \stackrel{\text{def}}{=} \{y \, , \; d(x, y) \le r\}$$

instead of balls $B(x, r)$. *Hidden solution:* *[UNACCESSIBLE UUID '0YY']*

E11.10 Prerequisites:10.a.8.Let $K \subseteq X$ compact; fix $\alpha > 1$; define $\tilde{d}(x,y) = \sqrt[\alpha]{d(x,y)}$. [0YZ]
We know from 10.a.8 that it is a distance. Show that

$$\alpha \dim(K,d) = \dim(K,\tilde{d}) \ .$$

In particular $K = [0,1]$ (the interval $K \subseteq X = \mathbb{R}$) with the distance $\tilde{d}(x,y) = \sqrt[\alpha]{|x-y|}$ has dimension α.

Hidden solution: *[UNACCESSIBLE UUID '0Z0']*

E11.11 Topics:norm.Prerequisites:12.10. [0Z1]

Let K be a compact in $\mathbb{R}^n$; we write $\dim(K, |\cdot|)$ to denote the limit that defines the dimension, using the balls of the Euclidean norm. Given a norm ϕ we can define the distance $d(x,y) = \phi(x-y)$, and with this calculate the dimension $\dim(K,\phi)$. Show that $\dim(K,|\cdot|) = \dim(K,\phi)$, in the sense that, if one limit exists, then the other limit exists, and they are equal. *Hidden solution:* *[UNACCESSIBLE UUID '0Z2']*

E11.12 We indicate an operating policy that can be used in the following exercises. [0Z3]

- If there is a descending sequence $\rho_j \to 0$ and h_j positive such that h_j balls of radious ρ_j are enough to cover K, then

$$\limsup_{\rho\to 0+} \frac{\log N(\rho)}{\log(1/\rho)} \le \limsup_{j\to\infty} \frac{\log h_{j+1}}{\log(1/\rho_j)} \ . \tag{11.13}$$

- If, on the other hand, there is a descending sequence $r_j \to 0$, and $C_n \subseteq K$ finite families of points that are at least distant r_j from each other, i.e. for which

$$\forall x,y \in C_n, x \ne y \Rightarrow d(x,y) \ge r_j \ , \tag{11.14}$$

then

$$\liminf_{\rho\to 0+} \frac{\log N(\rho)}{\log(1/\rho)} \ge \liminf_{j\to\infty} \frac{\log l_j}{\log(1/r_{j+1})} \ . \tag{11.15}$$

where $l_j = |C_j|$ is the cardinality of C_j. Note that the points of $x \in C_j$ are centers of disjoint balls $B(x, r_j/2)$, therefore $l_j \le P(r_j/2)$, as defined in 11.6.

In particular, if

$$\limsup_{j\to\infty} \frac{\log h_{j+1}}{\log(1/\rho_j)} = \liminf_{j\to\infty} \frac{\log l_j}{\log(1/r_{j+1})} = \beta \tag{11.16}$$

then the set K has dimension β.

Hidden solution: *[UNACCESSIBLE UUID '0Z5'][UNACCESSIBLE UUID '0Z6']*

E11.17 Prerequisites: 12.a.14 11.11 11.12.Difficulty:*.Let $K \subseteq \mathbb{R}^m$ compact. Consider [0Z7]
the family of closed cubes with edge length 2^{-n} and centers at the grid points $2^{-n}\mathbb{Z}^m$. We call it *"n-tessellation"*. Let N_n be the number of cubes of the *n-tessellation* intersecting K. Show that N_n is weakly increasing. Show that the following limit exists

$$\lim_{n\to\infty} \frac{\log_2 N_n}{n} \tag{11.18}$$

†84 By the Heine–Borel theorem 10.j.1 we know that $N(\rho) < \infty$

if and only if the limit (11.2) (that defines the dimension) exists. Show that, when they both exist, they coincide. This approach to computing the dimension is called *Box Dimension*.

Hidden solution: [UNACCESSIBLE UUID '0Z8'][UNACCESSIBLE UUID '0Z9']

These quantities have an interpretation in rate-distortion theory. "n" is the position of the last significant digit (in base 2) in determining the position of a point x. "$\log_2 N_n$" is the number of "bits" needed to identify any $x \in K$ with such precision.

E11.19 Let a_n be an infinitesimal decreasing positive sequence. Let $K \subseteq \mathbb{R}$ given by [0ZB]

$$K = \{0\} \cup \{a_n : n \in \mathbb{N}, n \geq 1\} ;$$

note that it is compact. Study the dimension of K in these cases:

- $a_n = n^{-\theta}$ with $\theta > 0$;
- $a_n = \theta^{-n}$ with $\theta > 1$.

Hidden solution: [UNACCESSIBLE UUID '0ZC']

E11.20 Let $1 \leq d \leq n$ be integers. Let $[0,1]^d$ be a cube of dimension d, we see it as a subset of $\mathbb{R}^n$ by defining [0ZD]

$$K = [0,1]^d \times \{(0, 0 \ldots 0)\}$$

namely

$$K = \{x \in \mathbb{R}^n, 0 \leq x_1 \leq 1, \ldots 0 \leq x_d \leq 1, x_{d+1} = \ldots = x_n = 0\}$$

Show that the dimension of K is d.

Hidden solution: [UNACCESSIBLE UUID '0ZF']

E11.21 Show that the dimension of the (image of) Koch curve is $\log 4 / \log 3$. (See for example [58] for the definition). [0ZG]

Hidden solution: [UNACCESSIBLE UUID '0ZH']

E11.22 Show that the dimension of the Cantor set is $\log(2)/\log(3)$. [0ZJ]

Hidden solution: [UNACCESSIBLE UUID '0ZK']

E11.23 Prerequisites:13.c.1,12.c.3.Inside the Banach space $X = C^0([0,a])$ endowed with the norm $\|\cdot\|_\infty$ we consider [0ZM]

$$K = \{f, f(0) = 0, \forall x, y, |f(x) - f(y)| \leq L|x - y|\}$$

where $L > 0, a > 0$ are fixed.

Estimate $N(\rho)$ for $\rho \to 0$

E11.24 Topics:ultrametric.Prerequisites:10.m.9. [0ZP]

Fix $\lambda > 0$. We define the ultrametric space of sequences as in Sec. §10.m.a: let I be a finite set, of cardinality p; let $X = I^{\mathbb{N}}$ be the space of sequences; define c as in eqn. (10.m.10); define $d(x,y) = \lambda^{-c(x,y)}$. We know from exercises 10.m.14 and 10.m.11 that (X,d) is compact.

Show that the dimension of (X,d) is $\log p / \log \lambda$.

Hidden solution: [UNACCESSIBLE UUID '0ZQ']

E11.25 Difficulty:*.Describe a compact set $K \subset \mathbb{R}$ for which the limit (11.2) does not exist. [0ZR]

§12 Normed spaces [0ZT]

Let in the following X be a vector space based on the real field $\mathbb{R}$.

Definition 12.1 (Norm). *A* norm *is an operation that maps a vector $v \in X$ in a real number $\|v\|$, which satisfies* [0ZV]

1. $\|v\| \geq 0$ *and* $\|v\| = 0$ *if and only if* $v = 0$*;*
2. *for every* $v \in X$ *and* $t \in \mathbb{R}$ *we have* $|t|\,\|v\| = \|tv\|$ *(we will say that the norm is* absolutely homogeneous*);*
3. (Triangle inequality) *for every* $v, w \in X$ *we have*

$$\|v + w\| \leq \|v\| + \|w\| \quad ;$$

this says that one side of a triangle is less than the sum of the other two.

Remark 12.2. *Many of the results in subsequent exercises generalize to the case of* "asymmetic norms"*, where the second request will be replaced by this:* for every real $t \geq 0$ you have $t\|v\| = \|tv\|$. *(In this case we will say that the norm is* positively homogeneous*).* [0ZW]

Exercises

E12.3 Let X be a vector space and $f : V \to \mathbb{R}$ a function that is *positively homogeneous*, that is: for every $v \in X$ and $t \geq 0$ you have $tf(v) = f(tv)$. [0ZX]

Show that f is convex if and only if the *triangle inequality* holds: for every $v, w \in X$ you have

$$f(v + w) \leq f(v) + f(w) \quad .$$

In particular, a norm is always a convex function.

E12.4 Note that if $v, w \in X$ are linearly dependent and have the same direction (i.e. you can write $v = \lambda w$ or $w = \lambda v$, for $\lambda \geq 0$), then you have [0ZY]

$$\|v + w\| = \|v\| + \|w\| \quad .$$

In particular, a norm is not a strictly convex function, because

$$\|v/2 + v/2\| = \frac{1}{2}\|v\| + \frac{1}{2}\|v\| \quad .$$

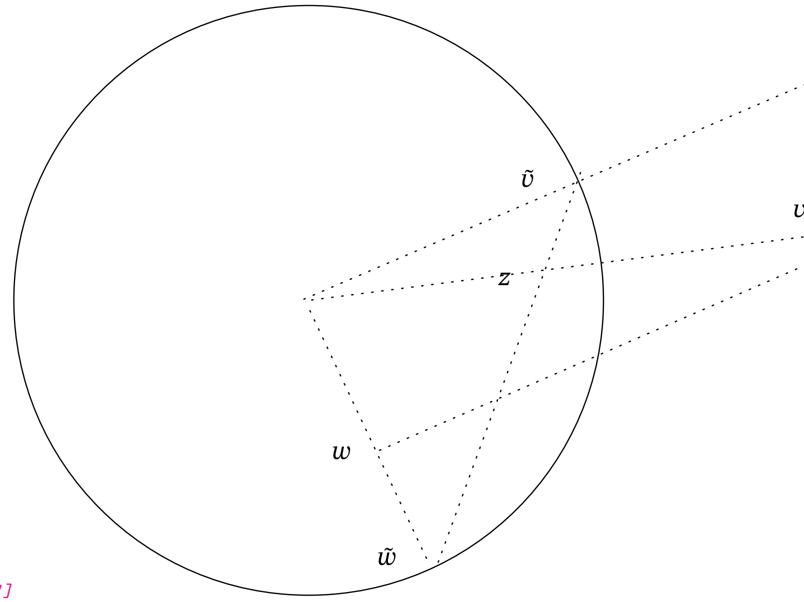
E12.5 Prerequisites:12.7, 15.d.10, 12.3.Difficulty:*.We will say that the normed space $(X, \|\cdot\|)$ is *strictly convex*[†85] if the following equivalent properties apply. [0ZZ]

- The disc $D = \{x \in X : \|x\| \leq 1\}$ is strictly convex. [†86]
- The sphere $\{x \in X, \|x\| = 1\}$ does not contain non-trivial segments (that is, segments of positive length).

[†85] See [31] for more properties.
[†86] The definition is in 15.d.10.

- For $v, w \in D$ with $\|v\| = \|w\| = 1$ and $v \neq w$, for every t such that $0 < t < 1$, we have that $\|tv + (1-t)w\| < 1$.
- For every $v, w \in X$ that are linearly independent we have $\|v + w\| < \|v\| + \|w\|$.

Show that the previous four clauses are equivalent.



Hidden solution: [UNACCESSIBLE UUID '102']

E12.6 Let X be a normed vector space with norm $\|\cdot\|$. Show that the sum operation $'+' : X \times X \to X$ is continuous. [105]

E12.7 Prerequisites:10.b.25. [106]

Let again X be a normed vector space with norm $\|\cdot\|$. Let $B(x,r) \stackrel{\text{def}}{=} \{y \in X : \|x-y\| < r\}$ be the ball. Let $D(x,r) \stackrel{\text{def}}{=} \{y \in X : \|x-y\| \leq r\}$ be the disk. Let $S(x,r) \stackrel{\text{def}}{=} \{y \in X : \|x-y\| = r\}$ be the sphere. Show that $\overline{B(x,r)} = D(x,r)$, that $B(x,r) = D(x,r)^\circ$, and that $\partial B(x,r) = \partial D(x,r) = S(x,r)$. Also show that $B(x,r)$ is not closed and $D(x,r)$ is not open.

E12.8 Prerequisites:10.b.21.Let X be a vector space, let ϕ, ψ be two norms on it. Show that the topologies generated by ϕ and ψ coincide, if and only if there exist $0 < a < b$ such that [107]

$$\forall x, \quad a\psi(x) \leq \phi(x) \leq b\psi(x) \ . \tag{12.9}$$

(When the relation (12.9) holds, we will say that the norms are *"equivalent"*).

Hidden solution: [UNACCESSIBLE UUID '108']

E12.10 *We want to show that "the norms in $\mathbb{R}^n$ are all equivalent."* [109]

Let $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ be the Euclidean norm. Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a norm: it can be shown that ϕ is a convex function, see 12.3; and therefore ϕ is a continuous function, see 15.b.9. Use this fact to prove that there exist $0 < a < b$ such that

$$\forall x, \quad a\|x\| \leq \phi(x) \leq b\|x\| \ . \tag{12.11}$$

Hidden solution: [UNACCESSIBLE UUID '10B']

§12.a Norms in Euclidean space [2CK]

Definition 12.a.1. *Given $p \in [1,\infty]$, the norms $\|x\|_p$ are defined on $\mathbb{R}^n$ with* [10C]

$$\|x\|_p = \begin{cases} \sqrt[p]{\sum_{i=1}^n |x_i|^p} & p \neq \infty \\ \max_{i=1}^n |x_i| & p = \infty \end{cases} \tag{12.a.2}$$

(The fact that these are norms is demonstrated by the 12.a.10).

Exercises

E12.a.3 Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$. [10D]

E12.a.4 Prerequisites:17.e.2.Having fixed $t, s \in [1,\infty]$ with $s > t$ and $x \in \mathbb{R}^n$, show that $\|x\|_s \le \|x\|_t$. Also show that $\|x\|_s < \|x\|_t$ if $n \ge 2$ and $x \neq 0$ and x is not a multiple of one of the vectors of the canonical basis $e_1, \dots e_n$. [10F]

Hints:

1. *use that $1 + t^p \le (1+t)^p$ for $p \ge 1$ and $t \ge 0$; or*
2. *use Lagrange multipliers; or*
3. *remember that $f(a+b) > f(a)+f(b)$ when $a \ge 0, b > 0$ $f(0) = 0$ and $f : [0,\infty) \to \mathbb{R}$ is strictly convex and continuous in 0 (see exercise 15.d.2), therefore derive $\frac{d}{dt}(\log \|x\|_t)$ and set $f(z) = z\log(z)$).*

Hidden solution: [UNACCESSIBLE UUID '10G']

E12.a.5 Having fixed $s, t \in [1,\infty]$ with $s < t$, show that $n^{-1/s}\|x\|_s \le n^{-1/t}\|x\|_t$ (where we agree that $n^{-1/\infty} = 1$). (Note that this is an inequality between averages). *(Hint. Set $\alpha = t/s$ and $y_i = |x_i|^s$, then use the convexity of $f(y) = y^\alpha$. Another tip: use 12.a.6.) Hidden solution:* [UNACCESSIBLE UUID '10K'] [10J]

E12.a.6 Let be given $p, q \in [1,\infty]$ such that $1/p + 1/q = 1$ [†87] and $x, y \in \mathbb{R}^n$; show the **Hölder inequality** in this form [10M]

$$\sum_{i=1}^n |x_i y_i| \le \|x\|_p \|y\|_q \quad . \tag{12.a.7}$$

In what cases is there equality?

Tips: Fix $x_i, y_i \ge 0$. For the cases with $p, q < \infty$ you can:

- *use Young inequality (15.d.3 or 24.16);*
- *use Lagrange multipliers;*
- *start from the case $n = 2$ and set $x_2 = tx_1$ and $y_2 = ay_1$; then, for cases $n \ge 3$ use induction.*

Hidden solution: [UNACCESSIBLE UUID '10N']

E12.a.8 Prerequisites:12.a.6.Infer the version [10P]

$$\sum_{i=1}^{n} x_i y_i \le \|x\|_p \|y\|_q \quad ; \tag{12.a.9}$$

from (12.a.7). In which case does equality apply?

E12.a.10 Prerequisites:12.a.6.Given $p \in [1, \infty]$ show the **Minkowski inequality** [10Q]

$$\|x + y\|_p \le \|x\|_p + \|y\|_p \quad . \tag{12.a.11}$$

There follows that $\|x\|_p$ are norms.

For $p \in (1, \infty)$ find a simple condition (necessary and sufficient) that involves equality; compare it with 12.4; deduce that $\mathbb{R}^n$, with the norm $\|\cdot\|_p$ for $p \in (1, \infty)$, is a *strictly convex normed space* (see 12.5). *Hidden solution:* [UNACCESSIBLE UUID '10R']

E12.a.12 Prerequisites:12.3,15.d.8,12.a.4.Let $r > 0$; if $p \in [1, \infty]$ then the ball $B_r^p = \{\|x\|_p < r\}$ is convex; also $B_r^p \subseteq B_r^{\tilde p}$ if $\tilde p > p$. In the case $n = 2$ of planar balls, study graphically the shape of the balls as p varies. Are there points that are on the border of all balls? *Hidden solution:* [UNACCESSIBLE UUID '10T'] [10S]

E12.a.13 If $r > 0$ and $p \in (1, \infty)$ then the sphere $\{\|x\|_p = r\}$ is a regular surface. *Hidden solution:* [UNACCESSIBLE UUID '10W'] [10V]

E12.a.14 Prerequisites:(12.a.2).We equip $\mathbb{R}^n$ with the norm $\|x\|_\infty$: show that in dimension 2 the disk $\{x \in \mathbb{R}^n, \|x\|_\infty \le 1\}$ is a square, and in dimension 3 it is a cube, *etc etc.* [10X]

Now we equip $\mathbb{R}^n$ with the norm $\|x\|_1$: show that in dimension 2 the disk $\{x \in \mathbb{R}^n, \|x\|_1 \le 1\}$ is a *rhombus* i.e. precisely a square rotated 45 degrees; and in dimension 3 the disk is an octahedron.

E12.a.15 Find a norm in $\mathbb{R}^2$ such that the ball is a regular polygon of n sides. [10Y]

Hidden solution: [UNACCESSIBLE UUID '10Z']

§12.b Isometries [2CH]

We rewrite the definition 10.i.1 in the case of normed spaces.

Definition 12.b.1. *If M_1, M_2 are vector spaces with norms $\|\|_{M_1}$ and respectively $\|\|_{M_2}$, then φ is an isometry when* [110]

$$\forall x, y \in M_1, \|x - y\|_{M_1} = \|\varphi(x) - \varphi(y)\|_{M_2} \tag{12.b.2}$$

(rewriting the definition of distance using norms).

We will compare it with this definition.

Definition 12.b.3. *Let B_1, B_2 be two normed vector spaces. A function $f : B_1 \to B_2$ is a **linear isometry** if it is linear and if* [111]

$$\|z\|_{B_1} = \|f(z)\|_{B_2} \; \forall\, z \in B_1 \quad . \tag{12.b.4}$$

†87 This means that if $p = 1$ then $q = \infty$; and vice versa.

If φ is linear then the definition of equation (12.b.2) is equivalent to the definition of *linear isometry* seen in equation (12.b.4) (just set $z = x - y$). This explains why both are called ”isometries”.

By the Mazur–Ulam theorem [60] if M_1, M_2 are vector spaces (on real field) equipped with norm and φ is a surjective isometry, then φ is affine (which means that $x \mapsto \varphi(x) - \varphi(0)$ is linear).

We now wonder if there are isometries that are not linear maps, or more generally affine maps.

Exercises

E12.b.5 Suppose the sphere $\{x \in M_2, \|x\|_{M_2} = 1\}$ contains no non-trivial segments: Then every function that satisfies (12.b.2) is necessarily affine. [112]

(See also Exercise 12.5.)

E12.b.6 The condition that φ is surjective cannot be removed from the Mazur–Ulam theorem. Find an example. [114]

Hint. By the previous exercise 12.b.5, the sphere $\{x \in M_2, \|x\|_{M_2} = 1\}$ *must contain segments.*

Hidden solution: *[UNACCESSIBLE UUID '115']*

§12.c Total convergence [2CJ]

Definition 12.c.1. *Let in the following X be a normed vector space based on the real field $\mathbb{R}$, with norm $\|\cdot\|$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. The series $\sum_{n=0}^{\infty} f_n$* converges totally *when $\sum_{n=0}^{\infty} \|f_n\| < \infty$.* [116]

Exercises

E12.c.2 Show that if the series of $(f_n)_n, (g_n)_n$ converge totally, then the series of $(f_n + g_n)_n$ converges totally. [117]

E12.c.3 `Topics:total convergence.Prerequisites:10.a.12,10.a.13,10.a.14.` [118]

Let V be a vector space with a norm $\|x\|$; So V is also a metric space with the metric $d(x, y) = \|x - y\|$. Show that the following two clauses are equivalent.

- (V, d) is complete.
- For each sequence $(v_n)_n \subset V$ such that $\sum_n \|v_n\| < \infty$, the series $\sum_n v_n$ converges.

(The second is sometimes called the ”total convergence criterion”)

A normed vector space $(V, |\cdot\|)$ such that the associated metric space (V, d) is complete, is called a **Banach space**.

Hidden solution: *[UNACCESSIBLE UUID '119']*

§12.d Norms of Linear application [2CM]

In the following $(X, \|\|_X)$and $(Y, \|\|_Y)$ will be normed spaces; $A : X \to Y$ is a linear application; we define the **induced norm** as

$$\|A\|_{X,Y} \stackrel{\text{def}}{=} \sup_{x\in X\ ,\ \|x\|_X \le 1} \|Ax\|_Y \ .$$

Exercises

E12.d.1 Show that $\|A\|_{X,Y} < \infty$ if and only if A is continuous. [11B]

E12.d.2 Note that if X has finite dimension then every linear application is continuous, and [11C]

$$\|A\|_{X,Y} = \max_{x\in X\ ,\ \|x\|_X \le 1} \|Ax\|_Y \ .$$

E12.d.3 Let $\mathcal{L}(X, Y)$ be the space of all continuous linear applications. Show that $\| \cdot \|_{X,Y}$ is a norm in $\mathcal{L}(X, Y)$. [11D]

E12.d.4 Let $(Z, \|\|_Z)$ be an additional normed space, and $B : Y \to Z$ a linear application. We similarly define [11F]

$$\|B\|_{Y,Z} \stackrel{\text{def}}{=} \sup_{y\in Y\ ,\ \|y\|_Y \le 1} \|By\|_Z \quad ;$$

show that

$$\|AB\|_{X,Z} \le \|A\|_{X,Y}\|B\|_{Y,Z} \ .$$

§12.e Norms of Matrixes [2CN]

Let then $p, q \in [1, \infty]$; we use the following norms $|x|_p$ defined in eqn. (12.a.2).

Definition 12.e.1. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix; considering it as a linear application between normed spaces $(\mathbb{R}^n, ||_p)$ and $(\mathbb{R}^m, ||_q)$, let's define again the* induced norm *as* [11G]

$$\|A\|_{p,q} \stackrel{def}{=} \max_{x\in\mathbb{R}^n\ ,\ |x|_p \le 1} |Ax|_q \tag{12.e.2}$$

(Note that the maximum is always reached at a point with $|x|_p = 1$).
The norm $\|A\|_{2,2}$ is called the spectral norm. .

Definition 12.e.3. *We also define the rules* [11H]

$$\|A\|_{F-\tilde p} = \begin{cases} \sqrt[\tilde p]{\sum_{i,j} |A_{i,j}|^{\tilde p}} & \tilde p < \infty \\ \max_{i,j} |A_{i,j}| & \tilde p = \infty \end{cases} ,$$

for $\tilde p \in [1, \infty]$. The case $\tilde p = 2$ is called Frobenious' norm.

Exercises

E12.e.4 Prerequisites:12.10.Note that the norms $\|A\|_{p,q}$ and $\|A\|_{F-\tilde{p}}$ are all equivalent. [11J]

E12.e.5 Prerequisites:12.d.4.Let's consider square matrices, i.e. $n = m$. We know from 12.d.4 that norms $\|A\|_{p,q}$ are submultiplicative, that is $\|AB\|_{p,q} \le \|A\|_{p,q}\|A\|_{p,q}$. [11K]

Show that the Frobenious norm is also submultiplicative.

Note that for a submultiplicative norm we have that $\|A^k\| \le \|A\|^k$ for every natural k.

E12.e.6 Show that [11M]

$$\|A\|_{1,1} = \max_{1\le j\le n} \sum_{i=1}^{m} |A_{i,j}|,$$

$$\|A\|_{\infty,\infty} = \max_{1\le i\le m} \sum_{j=1}^{n} |A_{i,j}| \ .$$

E12.e.7 If $A \in \mathbb{C}^{m\times n}$ we can define the induced norms [11N]

$$\|A\|_{p,q} \stackrel{\text{def}}{=} \max_{x\in\mathbb{C}^n \ , \ |x|_p\le 1} |Ax|_q \ . \tag{12.e.8}$$

Show that $\|A\|_{p,q} = \|\overline{A}\|_{p,q}$.

E12.e.9 Show that if $A \in \mathbb{R}^{m\times n}$ you have [11P]

$$\max_{x\in\mathbb{R}^n, |x|_2\le 1} |Ax|_2 = \max_{x\in\mathbb{C}^n, |x|_2\le 1} |Ax|_2 \ .$$

Hidden solution: [UNACCESSIBLE UUID '11Q']

§12.f Minkowski sum [2CP]

Let be in the following X be a vector space normed with norm $\|\cdot\|$.

Definition 12.f.1. *Let X be a vector space and $A, B \subseteq X$. We define the **Minkowski sum** $A \oplus B \subseteq X$ as* [11R]

$$A \oplus B = \{x + y \ : \ x \in A, y \in B\} \ .$$

In the following, given $A \subseteq X, z \in X$ we will indicate with $A+z = \{b+z \ : \ b \in B\}$ the translation of A in the direction z.

Exercises

E12.f.2 Prerequisites:12.f.1.Show that the sum is associative and commutative; and that the sum has a single neutral element, which is the set $\{0\}$ consisting of the origin alone. [11S]

E12.f.3 Prerequisites:12.f.1.If A is open, show that $A \oplus B$ is open. *Hidden solution:* [UNACCESSIBLE UUID '11V'] [11T]

E12.f.4 Prerequisites:12.f.1.If A, B are compact, show that $A \oplus B$ is compact. *Hidden solution:* [UNACCESSIBLE UUID '11X'] [11W]

E12.f.5 Prerequisites:12.f.1. If A is a closed set and B is a compact set, show that $A \oplus B$ is closed. *Hidden solution:* [UNACCESSIBLE UUID '11Z'] [11Y]

E12.f.6 Prerequisites:12.f.1. Show an example where A, B are closed but $A \oplus B$ is not closed. *Hidden solution:* [UNACCESSIBLE UUID '121'] [120]

E12.f.7 Prerequisites:12.f.1. If A, B are convex show that $A \oplus B$ is convex. *Hidden solution:* [UNACCESSIBLE UUID '123'] [122]

See also the exercises 6.c.12 and 10.d.4.

§12.g Mathematical morphology [2CQ]

Let be in the following X be a vector space normed with norm $\|\cdot\|$.

Definition 12.g.1. *For $A, B \subseteq X$ arbitrary subsets, we recall the definition of* Minkowski sum $A \oplus B = \{x + y : x \in A, y \in B\}$ *defined in 12.f.1.* [124]

Having now fixed a set B, we define

- *the **dilation** of a set $A \subseteq X$ to be $A \oplus B$;*
- *the **erosion** of a set $A \subseteq X$ as*

$$A \ominus B = \{z \in X : (B + z) \subseteq A\} \quad ;$$

- *the **closing** $A \bullet B = (A \oplus B) \ominus B$;*
- *the **opening** $A \circ B = (A \ominus B) \oplus B$.*

Where, given $B \subseteq X, z \in X$, we have indicated with $B + z = \{b + z : b \in B\}$ the translation of B in the direction z. In previous operations B it is known as "structural element", And in applications often B it's a puck or a ball.

Let in the following $A, B, C \subseteq X, w, z \in X$. Some of the following exercises are taken from [27].

Exercises

E12.g.2 Prerequisites:12.g.1. [125]

Show the following identities:

$$A \oplus B = \bigcup_{y \in B} (A + y)$$
$$A \ominus B = \bigcap_{y \in B} (A - y)$$

Hidden solution: [UNACCESSIBLE UUID '126']

E12.g.3 Prerequisites:12.g.2,12.g.1. [127]

Let $\tilde{B} = \{-b : b \in B\}$: show that $(A \oplus B)^c = A^c \ominus \tilde{B}$, where $A^c = X \setminus A$ is the complementary. *Hidden solution:* [UNACCESSIBLE UUID '128']

E12.g.4 Prerequisites:12.g.1,12.g.2. [129]

Show that the four operations are monotonic: if $A \subseteq C$ then $A \oplus B \subseteq C \oplus B$, $A \ominus B \subseteq C \ominus B$, $A \bullet B \subseteq C \bullet B$ and $A \circ B \subseteq C \circ B$. *Hidden solution:* *[UNACCESSIBLE UUID '12B']*

E12.g.5 Prerequisites:12.g.1,12.f.3,12.g.3.If A is closed, show that $A \ominus B$ is closed. *Hidden solution:* *[UNACCESSIBLE UUID '12D']* [12C]

E12.g.6 Prerequisites:12.g.1. [12F]

Show that erosion has the invariant property in this sense:

$$(A + z) \ominus (B + z) = A \ominus B\ .$$

E12.g.7 Prerequisites:12.g.1. [12G]

Moreover, the erosion satisfies $(A \ominus B) \ominus C = A \ominus (B \oplus C)$. *Hidden solution:* *[UNACCESSIBLE UUID '12H']*

E12.g.8 Prerequisites:12.g.1. [12J]

Show that the expansion enjoys the distributive property with respect to union:

$$(A \cup C) \oplus B = (A \oplus B) \cup (C \oplus B).$$

Hidden solution: *[UNACCESSIBLE UUID '12K']*

E12.g.9 Prerequisites:12.g.1,12.g.8,12.g.3.Show that erosion has the distributive property with respect to the intersection: [12M]

$$(A \cap C) \ominus B = (A \ominus B) \cap (C \ominus B).$$

Hidden solution: *[UNACCESSIBLE UUID '12N']*

E12.g.10 Prerequisites:12.g.1,12.g.3.Sia $\tilde{B} = \{-b\ :\ b \in B\}$. Show that [12P]

$$(A \bullet B)^c = (A^c \circ \tilde{B})\ .$$

Hidden solution: *[UNACCESSIBLE UUID '12Q']*

E12.g.11 Prerequisites:12.g.1. [12R]

Show that

$$A \subseteq (C \ominus B)$$

if and only if

$$(A \oplus B) \subseteq C\ \ .$$

Hidden solution: *[UNACCESSIBLE UUID '12S']*

E12.g.12 Prerequisites:12.g.1. [12T]

Recall that the operation $A \bullet B = (A \oplus B) \ominus B$ is called ”closing”.

- Show that $A \subseteq A \bullet B$.
- Let $X = \mathbb{R}^n$, $B = B_r = \{\|x\| < r\}$ a ball, find an example of a set A that is open non-empty bounded, and $A \bullet B = A$.

- Setting $X = \mathbb{R}^n$, $B = B_r$ a ball, find an example where $A \bullet B \neq A$.

Hidden solution: [UNACCESSIBLE UUID '12V']

E12.g.13 Prerequisites:12.g.1,12.g.2. [12W]

The opening is also given by $A \circ B = \bigcup_{x \in X, B+x \subseteq A}(B+x)$, which means that it is the locus of translations of the structuring element B inside the set A. *Hidden solution:* [UNACCESSIBLE UUID '12X']

E12.g.14 Prerequisites:12.g.1.In the following $A, B, \hat{B} \subseteq \mathbb{R}^n$. [12Y]

- Find an example where $B \overset{\neq}{\subset} \hat{B}$ and $A \circ B \overset{\neq}{\subset} A \circ \hat{B}$.
- Find an example where $B \overset{\neq}{\subset} \hat{B}$ and $A \circ \hat{B} \overset{\neq}{\subset} A \circ B$.

Hidden solution: [UNACCESSIBLE UUID '12Z']

E12.g.15 Prerequisites:12.g.1,12.g.13.If A is convex and $\hat{B}$ is the convex envelope (see 15.a.15 of B, show that $A \circ B \subseteq A \circ \hat{B}$. Show with an example that equality may not apply. *Hidden solution:* [UNACCESSIBLE UUID '131'] [130]

E12.g.16 Prerequisites:12.g.1,12.g.13.If A, B are convex, show that $A \circ B$ is convex. *Hidden solution:* [UNACCESSIBLE UUID '133'] [132]

§13 Semicontinuity, right and left limits [137]

§13.a Semi continuity [2CV]

Let (X,τ) be a topological space.

Definition 13.a.1. *A function $f : X \to \mathbb{R}$ is said* lower semicontinuous *(abbreviated* l.s.c.*) if* [138]

$$\forall x_0 \in D(X) \quad , \quad \liminf_{x\to x_0} f(x) \ge f(x_0)$$

and vice versa it says upper semicontinuous *(abbreviated* u.s.c.*) if*

$$\forall x_0 \in D(X) \quad , \quad \limsup_{x\to x_0} f(x) \le f(x_0) \; .$$

($D(X)$ are the accumulation points in X).

Note that f is *lower semi continue* if and only if $(-f)$ is *upper semi continue*: so in many subsequent exercises we will only see cases *l.s.c.* cases.

Exercises

E13.a.2 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 1$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(0) = 0$, and $f(x) = 1/q$ if $|x| = p/q$ with p, q coprime integers, $q \ge 1$. Show that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous in every $t \in \mathbb{Q}$. [139]

Show that the described function is u.s.c. *Hidden solution:* *[UNACCESSIBLE UUID '13B']*

E13.a.3 `Prerequisites:13.a.2.` [13C]

Construct a monotonic function with the same property as the one seen in the exercise 13.a.2.

E13.a.4 Let $f : X \to \mathbb{R}$; the following assertions are equivalent. [13D]

1. f is *lower semicontinuous*.
2. For every t, we have that the sublevel

   $$S_t = \{x \in X, f(x) \le t\}$$

   is closed.
3. The epigraph

   $$E = \{(x,t) \in X \times \mathbb{R}, f(x) \le t\}$$

   is closed in $X \times \mathbb{R}$.

Note that the second condition means that f is continuous from (X,τ) to $\mathbb{R}, \tau_+$ where $\tau_+ = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is the set of half-lines, which is a topology (easy verification).

Then formulate the equivalent theorem for functions *upper semicontinuous*.

Hidden solution: *[UNACCESSIBLE UUID '13F']*

E13.a.5 If $f, g : X \to \mathbb{R}$ are lower semicontinuous, then $f+g$ is l.s.c. *Hidden solution:* *[UNACCESSIBLE UUID '13H']* [13G]

E13.a.6 Let I be a family of indices. Supposte that, for $n \in I$, $f_n : X \to \mathbb{R}$ are l.s.c. functions. We define $f \stackrel{\text{def}}{=} \sup_{n\in I} f_n$, then f is l.s.c. (defined as $f : X \to \mathbb{R}\cup\{+\infty\}$). [†88]. *Hidden solution:* [UNACCESSIBLE UUID '13K'] [13J]

E13.a.7 Vice versa, given $f : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ l.s.c., there exists an increasing sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to_n f(x)$. *Hidden solution:* [UNACCESSIBLE UUID '13N'] [13M]

E13.a.8 `Topics:inf-convolution.Difficulty:*`. When (X, d) is a metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and bounded from below, let [13P]

$$f_n(x) \stackrel{\text{def}}{=} \inf_{y\in X} \{f(y) + nd(x, y)\}$$

be the *inf-convolution*. Show that the sequence f_n is an increasing sequence of Lipschitz functions with $f_n(x) \to_n f(x)$. *Hidden solution:* [UNACCESSIBLE UUID '13Q']

E13.a.9 Given $f : X \to \mathbb{R}$, define [13R]

$$f^*(x) = f(x) \vee \limsup_{y\to x} f(y) \quad ;$$

show that $f^*(x)$ is the smallest upper semicontinuous function that is greater than or equal to f at each point.

Similarly, define

$$f_*(x) = f(x) \wedge \liminf_{y\to x} f(y)$$

then $-(f^*) = (-f)_*$, and therefore $f_*(x)$ is the greatest lower semicontinuous function that is less than or equal to f at each point.

Finally, note that $f^* \geq f_*$.

Hidden solution: [UNACCESSIBLE UUID '13S']

E13.a.10 `Topics:oscillation`. [13T]

Given any $f : X \to \mathbb{R}$, we define *oscillation function* $\text{osc}(f)$

$$\text{osc}(f)(x) \stackrel{\text{def}}{=} f^*(x) - f_*(x)$$

1. Note that $\text{osc}(f) \geq 0$, and that f is continuous in x if and only if $\text{osc}(f)(x) = 0$.
2. Show that $\text{osc}(f)$ is upper semicontinuous.
3. If (X, d) is a metric space, note that

$$\text{osc}(f)(x) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0+} \sup\{|f(y) - f(z)| \ ,\ d(x, y) < \varepsilon, d(x, z) < \varepsilon\} \quad .$$

Hidden solution: [UNACCESSIBLE UUID '13V']

E13.a.11 Let (X, τ) be a topological space and $f : X \to \mathbb{R}$ a function. Let $\overline{x} \in X$ be an accumulation point. Let eventually U_n be a family of open neighbourhoods of $\overline{x}$ [13W]

[†88]Note that this is also true when $n \in I$ is an uncountable family of indices; and it is also true when f_n are continuous

with $U_n \supseteq U_{n+1}$. Then there exists a sequence $(x_n) \subset X$ with $x_n \in U_n$ and $x_n \neq \overline{x}$ and such that

$$\lim_{n\to\infty} f(x_n) = \liminf_{x\to\overline{x}} f(x) \quad .$$

(Note that in general we do not claim neither expect that $x_n \to \overline{x}$). *Hidden solution:* [UNACCESSIBLE UUID '13X']

E13.a.12 Let (X, τ) be a topological space and $f : X \to \mathbb{R}$ a function; let $\overline{x} \in X$ be an accumulation point; let A be the set of all the limits $\lim_n f(x_n)$ (when they exist) for all sequences $(x_n) \subset X$ such that $x_n \to \overline{x}$; then [13Y]

$$\liminf_{x\to\overline{x}} f(x) \le \inf A \quad ;$$

moreover, if (X, τ) satisfies the first axiom of countability, then equality holds and $\inf A = \min A$.

E13.a.13 Let $f_1 : [0,\infty] \to [0,\infty]$ monotonic function (weakly increasing) and right continuous. Let then $f_2 : [0,\infty) \to [0,\infty]$ be given by [13Z]

$$f_2(s) = \sup\{t \ge 0 : f_1(t) > s\}$$

(with the convention that $\sup \emptyset = 0$) and then again $f_3 : [0,\infty) \to [0,\infty]$ defined by

$$f_3(s) = \sup\{t \ge 0 : f_2(t) > s\} \quad :$$

then $f_1 \equiv f_3$.

Hidden solution: [UNACCESSIBLE UUID '140']

§13.b Regulated functions [2CT]

Definition 13.b.1. [141]

Let $I \subset \mathbb{R}$ be an interval. ***Regulated functions*** *$f : I \to \mathbb{R}$ are the functions that admit, at every point, right and left limits.* †89

(Note in particular that every monotonic function is regulated, and every continuous function is regulated.)

Exercises

E13.b.2 Show that a regulated function $f : [a,b] \to \mathbb{R}$ is bounded. [142]

E13.b.3 Prerequisites:13.a.10. Let $I = [a,b]$ be closed and bounded interval. Show that [143]

- $f : [a,b] \to \mathbb{R}$ is regulated if and only if
- for any $\varepsilon > 0$, there exists a finite set of points $P \subset I$ such that, for every $J \subseteq I$ with J an open interval that does not contain any point of P, the oscillation of f in J is less than ε.

E13.b.4 Let $I = [a, b]$. Let V be the set of functions $f : [a, b] \to \mathbb{R}$ that are piecewise constant; it is the vector space generated by $\mathbb{1}_J$, all the characteristic functions of all intervals $J \subseteq I$. Prove that the closure of V (according to uniform convergence) coincides with the space of regulated functions. [144]

So the space of regulated functions, endowed with the norm $\|\cdot\|_\infty$, is a Banach space.

See also exercises 16.7, 16.8, 16.9 and 18.8.

§13.c Sup transform [2CR]

Definition 13.c.1. *Suppose that either $I = \mathbb{R}^+$ or $I = \mathbb{R}$ in the following, for simplicity.* [2CS]

Let $\varepsilon > 0$; given a bounded function $f : I \to \mathbb{R}$ [†90] *, we define the "sup transform" as the function $g : I \to \mathbb{R}$ given by*

$$g(x) = \sup_{y \in (x, x+\varepsilon)} f(y) \ . \qquad (13.c.2)$$

We summarize this transformation with the notation $g = F(\varepsilon, f)$.

Exercises

E13.c.3 Prerequisites:13.c.1.Show that g is regulated. [145]

E13.c.4 Prerequisites:13.c.1.Show that g is lower semicontinuous. [146]

E13.c.5 Prerequisites:13.c.1.Show that $f = g$ if and only if f is monotonic weakly decreasing and right continuous. [147]

E13.c.6 Prerequisites:13.c.1.Given [148]

$$g(x) = \begin{cases} -1 & x = 4 \\ 0 & x \neq 4 \end{cases}$$

find f such that $g = F(1, f)$.

Hidden solution: *[UNACCESSIBLE UUID '149']*

E13.c.7 Prerequisites:13.c.1.Show that if f is continuous then g is continuous. [14B]

Hidden solution: *[UNACCESSIBLE UUID '14C']*

E13.c.8 Prerequisites:13.c.1,12.c.3.Let $C_b = C_b(I)$ be the space of continuous bounded functions $f : I :\to \mathbb{R}$). This is a Banach space (a complete normed space) with the norm $\|f\|_\infty = \sup_x |f(x)|$. [14D]

Consider the map $F : [0, \infty) \times C_b \to C_b$ transforming $g = F(\varepsilon, f)$, as defined in the eqn. (13.c.2).

Show that F is continuous.

E13.c.9 Prerequisites:13.c.1.How do previous exercises change if you define instead [14F]

$$g(x) = \sup_{y \in [x, x+\varepsilon]} f(y) \ ? \qquad (13.c.10)$$

Hidden solution: *[UNACCESSIBLE UUID '14G']*

[†89] At the extremes, of course, only one of the two limits is required.

[†90] The "bounded" hypothesis is convenient, the following resulst are valid even without this hypothesis, with simple modifications.

§14 Continuity [14J]

§14.a Continuous functions [2DP]

Definition 14.a.1. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function; let $x \in A$; f is called* ***continuous at*** *x if* [2DN]

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in A,\ |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon \ .$$

f is called ***continuous*** *if it is continuous in every point.*

The set of all continuous functions $f : A \to \mathbb{R}$ is denoted with $C(A)$; it is a vector space.

Further informations on this subject may be found in Chap. 3 in [4], or Chap. 4 of [22].

Exercises

E14.a.2 Suppose that $f : (0, 1] \to \mathbb{R}$ is a continuous function. Prove that, it is bounded from above †91 if and only if $\limsup_{x\to 0+} f(x) < +\infty$. [14K]

E14.a.3 `Prerequisites:10.g.8.`Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Let it be shown that there at most countably many points where a discontinuity may be removed (i.e. the points z for which $\lim_{x\to z} f(x) \neq f(z)$, see [52]). [14M]

E14.a.4 `Prerequisites:10.g.8.`Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Show that the set of discontinuity points of the second type is countable at most (i.e. the points z where the lateral limits exist but $\lim_{x\to z+} f(x) \neq \lim_{x\to z-} f(x)$, see [52]). [14N]

E14.a.5 `Prerequisites:6.8.`Fixed $\alpha > 1$ we define, for $x \in \mathbb{R}$, α^x as in 6.8. Show that this is a continuous function and that it is a homeomorphism between $\mathbb{R}$ and $(0, \infty)$. The inverse of $y = \alpha^x$ is the function **logarithm** $x = \log_\alpha y$. [21N]

E14.a.6 `Prerequisites:10.f.4.Difficulty:*.` [14P]

Let $C \subset \mathbb{R}$ be a closed set, and let $f : C \to \mathbb{R}$ be continuous function. Show that there exists $g : \mathbb{R} \to \mathbb{R}$ continuous and extending f, i.e. $g_{|C} = f$.

Hidden solution: *[UNACCESSIBLE UUID '14Q']*

E14.a.7 `Difficulty:**.`Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not monotonic in any interval (open nonempty). [14R]

E14.a.8 `Prerequisites:Riemann integral.` [14T]

Given a continuous function $f = f(x, y) : \mathbb{R} \times [0, 1] \to \mathbb{R}$, setting

$$g(x) = \int_0^1 f(x, y) \mathrm{d}y \quad ,$$

show that g is continuous.

Hidden solution: *[UNACCESSIBLE UUID '14V']*

E14.a.9 Given a continuous function $f = f(x, y) : \mathbb{R} \times [0, 1] \to \mathbb{R}$, and setting [14W]

$$g(x) = \max_{y\in[0,1]} f(x, y)$$

show that g is continuous. *Hidden solution:* *[UNACCESSIBLE UUID '14X']*

E14.a.10 Let x_n, y_n be strictly positive real sequences with limit zero; there is a continuous and monotonic function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $\forall x > 0, f(x) > 0$, and such that $\forall n, f(x_n) < y_n$ (hence $\lim_{x\to 0+} f(x) = 0$). [14Y]

Hidden solution: *[UNACCESSIBLE UUID '14Z']*

E14.a.11 Let be given a function $g : [0, \infty) \to [0, \infty]$ such that $g(0) = 0$ and $\lim_{x\to 0+} g(x) = 0$; then there exists a continuous and monotonic function $f : [0, \infty) \to [0, \infty]$ such that $f(0) = 0$, $\lim_{x\to 0+} f(x) = 0$, and $f \ge g$. [150]

E14.a.12 Prove that if a monotonic function is defined on a dense subset of an open interval I, and has dense image in another open interval J, then it can be extended to a monotonic continuous function between the two open intervals I, J. [151]

(What happens if I is closed but J is open?)

E14.a.13 `Prerequisites:categories of Baire Sec. §10.k.Difficulty:*.` [152]

Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on the rational points and discontinuous on the irrational points. *(Hint. Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is not a F_σ set in $\mathbb{R}$, using Baire's theorem.)*

Hidden solution: *[UNACCESSIBLE UUID '153']*

§14.b Uniformly continuous functions [2DQ]

Definition 14.b.1. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function; f is called* ***uniformly continuous*** *if* [155]

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in A,\ |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon\ .$$

More in general, given (X_1, d_1) and (X_2, d_2) metric spaces, given the function $f : X_1 \to X_2$, f is ***uniformly continuous*** *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in X_1,\ d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon\ .$$

It is easy to see that a function *uniformly continuous* is continuous at every point.

Exercises

E14.b.2 `Prerequisites:14.b.1.`Let $f : X_1 \to X_2$ with (X_1, d_1) and (X_2, d_2) metric spaces. [156]

A monotonic (weakly) increasing function $\omega : [0, \infty) \to [0, \infty]$, with $\omega(0) = 0$ and $\lim_{t\to 0+} \omega(t) = 0$, such that

$$\forall x, y \in X_1,\ \ d_2(f(x), f(y)) \le \omega(d_1(x, y))\ , \tag{14.b.3}$$

[†91] i.e. there exists $c \in \mathbb{R}$ such that $\forall x \in (0, 1]$ you have $f(x) < c$

is called **continuity modulus** for the function f. (Note that f can have many continuity moduli).

For example, if the function f is Lipschitz, i.e. there exists $L > 0$ such that

$$\forall x, y \in X_1, \quad d_2(f(x), f(y)) \le L\, d_1(x, y)$$

then f satisfies the eqz. (14.b.3) by placing $\omega(t) = Lt$.

We will now see that the existence of a continuity modulus is equivalent to the uniform continuity of f.

- If f is uniformly continuous, show that the function

$$\omega_f(t) = \sup\{d_2(f(x), f(y)) \;:\; x, y \in X_1, d_1(x, y) \le t\} \qquad (14.b.4)$$

  is the smallest continuity modulus.[†92]
- Note that the modulus defined in (14.b.4) may not be continuous, and may be infinite for t large — find examples of this behaviour.
- Also show that if f is uniformly continuous, there is a modulus that is continuous where it is finite.
- Conversely, it is easy to verify that if f has a continuity modulus, then it is uniformly continuous.

If you don't know metric space theory, you can prove the previous results in case $f : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$. (See also the exercise 14.b.12, which shows that in this case the modulus ω defined in (14.b.4) is continuous and is finite).

Hidden solution: *[UNACCESSIBLE UUID '157'] [UNACCESSIBLE UUID '158'] [UNACCESSIBLE UUID '159']*

E14.b.5 Let (X, d) metric space and $\mathcal{F}$ the set of uniformly continuous functions $f : X \to \mathbb{R}$, show that $\mathcal{F}$ is a vector space. [15C]

This is more generally true if $f : X \to X_2$ where X_2 is a normed vector space (to which we associate the distance derived from the norm).

Hidden solution: *[UNACCESSIBLE UUID '15D']*

E14.b.6 Difficulty:*.Let (X_1, d_1) and (X_2, d_2) metric spaces, with (X_2, d_2) complete. Let $A \subset X_1$ and $f : A \to X_2$ be a uniformly continuous function. Show that there is a uniformly continuous function $g : \overline{A} \to X_2$ extending f; In addition, the extension g is unique. [15F]

Note that if ω is a continuity modulus for f then it is also a continuity modulus for g. (We assume that ω is continuous, or, at least, that it is upper semicontinuous).
Hidden solution: *[UNACCESSIBLE UUID '15G'] [UNACCESSIBLE UUID '15H']*

E14.b.7 Prerequisites:14.b.6.Let $A \subset \mathbb{R}^n$ be bounded and $f : A \to \mathbb{R}$ a continuous function. Show that f is uniformly continuous if and only there exists a continuous function $g : \overline{A} \to \mathbb{R}$ extending f; In addition, the extension g is unique. [15J]

Hidden solution: *[UNACCESSIBLE UUID '15K']*

E14.b.8 Let $f : (0, 1] \to \mathbb{R}$ be a continuous function. Prove that it is uniformly continuous, if and only if the limit $\lim_{x\to 0+} f(x)$ exists and is finite. *Hidden solution:* *[UNACCESSIBLE UUID '15N']* [15M]

E14.b.9 Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function and such that the limit $\lim_{x\to\infty} f(x)$ exists and is finite. Show that it is uniformly continuous. *Hidden solution:* *[UNACCESSIBLE UUID '15Q']* [15P]

E14.b.10 Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function, show that these two clauses are equivalent. [15R]

- There exists $g : [0, \infty) \to \mathbb{R}$ uniformly continuous and such that the limit $\lim_{x\to\infty}(f(x) - g(x))$ exists and finite.
- f is uniformly continuous.

Hidden solution: *[UNACCESSIBLE UUID '15S']*

E14.b.11 Find an example of $f : [0, \infty) \to \mathbb{R}$ continuous and bounded, but not uniformly continuous. *Hidden solution:* *[UNACCESSIBLE UUID '15V']* [15T]

E14.b.12 Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be uniformly continuous. Let ω be the continuity modulus, defined by the eqz. (14.b.4), as in the exercise 14.b.2. Show that ω is subadditive i.e. [15W]

$$\omega(t) + \omega(s) \ge \omega(t + s) \quad .$$

Knowing that $\lim_{t\to 0+} \omega(t) = 0$ we conclude that ω is continuous. *Hidden solution:* *[UNACCESSIBLE UUID '15X']*

E14.b.13 Prerequisites:14.b.12. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous; show that [15Z]

$$\limsup_{x\to\pm\infty} |f(x)|/x < \infty$$

or, equivalently, that there exists a constant C such that $|f(x)| \le C(1 + |x|)$ for every x. *Hidden solution:* *[UNACCESSIBLE UUID '160']*

E14.b.14 Prerequisites:10.b.22. Let (X_1, d_1), (X_2, d_2) and (Y, δ) be three metric spaces; consider the product $X = X_1 \times X_2$ equipped with the distance $d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$. [†93] Let $f : X \to Y$ be a function with the following properties: [161]

- For each fixed $x_1 \in X_1$ the function $x_2 \mapsto f(x_1, x_2)$ is continuous (as a function from X_2 to Y);
- There is a continuity modulus ω such that

  $$\forall x_1, \tilde{x}_1 \in X_2 \ , \ \forall x_2 \in X_2 \ , \delta\big(f(x_1, x_2), f(\tilde{x}_1, x_2)\big) \le \omega\big(d_1(x_1, \tilde{x}_1)\big)$$

  (We could define this property by saying that the function $x_1 \mapsto f(x_1, x_2)$ is uniformly continuous, with constants independent of the choice of x_2).

Then show that f is continuous.

See also point 3 of the exercise 18.8.

§14.c Lipschitz and Hölder functions [2DR]

Definition 14.c.1. *Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said **Lipschitz continuous** if there exists $L > 0$ such that $\forall x, y \in A$,* [162]

$$|f(x) - f(y)| \le L|x - y| \ .$$

*A function $f : A \to \mathbb{R}$ is said **Hölder continuous** if $L > 0$ and $\alpha \in (0, 1]$ exist such that $\forall x, y \in A$,*

$$|f(x) - f(y)| \le L|x - y|^\alpha \ .$$

The constant α is called the order.

As in the case of "uniform continuity", this notion extends to maps between metric spaces.

Exercises

E14.c.2 `Prerequisites:14.b.2.` Show that the Lipschitz functions, as well as Hölder functions, are uniformly continuous What can be said about their continuity modulus? [163]

E14.c.3 Let $I \subset \mathbb{R}$ be an open interval. Let $f : I \to \mathbb{R}$ be differentiable. Show that f' is bounded on I, if and only if f is Lipschitz continuous. [164]

E14.c.4 Let $I \subset \mathbb{R}$ interval. Let $f : I \to \mathbb{R}$ such that there exists $\alpha > 1$ such that $\forall x, y, |f(x) - f(y)| \le |x - y|^\alpha$ (i.e. f is Hölder continuous of order $\alpha > 1$): Show that f is constant. [165]

E14.c.5 Let be given $f : [a, b] \to \mathbb{R}$ and a decomposition of $[a, b]$ into intervals $I_1 = [a, t_1], I_2 = [t_1, t_2], \ldots, I_n = [t_{n-1}, b]$ such that the restriction of f on each I_k is Lipschitz of constant C. Show that f is Lipschitz of constant C. [166]

Similarly for Hölder functions.

E14.c.6 Let $f : [a, b] \to \mathbb{R}$ Hölder with exponent $\alpha \le 1$. Show that f is Hölderian with exponent β for every $\beta < \alpha$. [167]

Note that this is not technically true for $f : \mathbb{R} \to \mathbb{R}$.

E14.c.7 Build $f : [0, 1] \to \mathbb{R}$ that is continuous but not Hölder continuous. *Hidden solution:* *[UNACCESSIBLE UUID '16B'] [UNACCESSIBLE UUID '16C']* [169]

E14.c.8 A linear function $f : \mathbb{R}^n \to \mathbb{R}^k$ is Lipschitz. [16D]

E14.c.9 For each of the following functions, say if it is continuous, uniformly continuous, Hölder (and with which exponent), or Lipschitz. [16F]

- $f : (0, 1) \to \mathbb{R}$, $f(x) = \sin(1/x)$.
- $f : (0, 1) \to \mathbb{R}$, $f(x) = x^{1/x}$.
- $f : (1, \infty) \to \mathbb{R}$, $f(x) = \sin(x^2)/x$

†92 Note that the family on which the upper bound is calculated always contains the cases $x = y$, therefore $\omega(t) \ge 0$.

†93 We know from 12.10 and 10.b.22 that there are several possible choices of distances, but they are equivalent to each other.

- $f : [-1,1] \to \mathbb{R}$, $f(x) = |x|^\beta$ with $\beta > 0$.
- $f : (0,\infty) \to \mathbb{R}$, $f(x) = \sin(x^\beta)$ with $\beta > 0$.

Hidden solution: [UNACCESSIBLE UUID '16H']

E14.c.10 Given $L \in (0,1)$ if $f : \mathbb{R} \to \mathbb{R}$ satisfies [16J]

$$|f(x) - f(y)| \le L|x - y| \quad \forall x, y \in \mathbb{R}$$

Then there is only one ”fixed point” that is a point x for which $f(x) = x$.

E14.c.11 Find a function $f : \mathbb{R} \to \mathbb{R}$ such that [16K]

$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in \mathbb{R}$$

but for which there is no ”fixed point” (that is a point x for which $f(x) = x$=.

Hidden solution: [UNACCESSIBLE UUID '16M']

§14.d Discontinuous functions [2DS]

Let be in the following (X, d) a metric space.

Definition 14.d.1. *A set E is called a F_σ if it is a countable union of closed sets. (See also exercise 10.b.30).* [2CX]

Exercises

E14.d.2 Note that every open set $A \subset X$ nonempty is a F_σ set. *(Hint: use 10.d.3).* [16N]
Hidden solution: [UNACCESSIBLE UUID '16P']

E14.d.3 `Prerequisites:13.a.10,13.a.4.`Given a generic $f : X \to \mathbb{R}$, show that the set E of points where f is discontinuous is a F_σ. *Hidden solution:* [UNACCESSIBLE UUID '16R'] [16Q]

E14.d.4 `Prerequisites:14.d.1.Difficulty:*.` [16S]

Suppose (X, d) admits a subset D that is dense but has empty interior. [†94]

Given a $E \subset X$ which is a F_σ, construct a function $f : X \to \mathbb{R}$ for which E is the set of points of discontinuity.

Hidden solution: [UNACCESSIBLE UUID '16T']

[†94] That is, both D and the complement $X \setminus D$ are dense. $X = \mathbb{R}$ meets this requirement, taking as an example $D = \mathbb{Q}$.

§15 Convex functions and sets [16V]

We will now discuss convexity. For simplicity, all results are presented using $\mathbb{R}^n$ as domain; but most results hold more in general in a generic vector space.

§15.a Convex sets [2F0]

Definition 15.a.1. *Given $x_1, \dots x_k \in \mathbb{R}^n$, given $t_1, \dots t_k \geq 0$ with $t_1 + \dots + t_k = 1$, the sum* [16W]

$$x_1 t_1 + \dots + x_k t_k$$

is a convex combination *of the points $x_1, \dots x_k$.*

Remark 15.a.2. *If $k = 2$ then the convex combination is usually written as $(tx + (1 - t)y)$ with $t \in [0, 1]$; the set of all these points is the segment that connects x to y.* [23P]

Definition 15.a.3. *Let $C \subseteq \mathbb{R}^n$ be a set; it is called* convex *if* [16X]

$$\forall t \in [0, 1], \quad \forall x, y \in C, \quad (tx + (1 - t)y) \in C$$

that is, if the segment connecting each $x, y \in C$ is all inclusive in C.

(We note that $\emptyset$ is a convex set, and that every vector subspace or affine subspace of $\mathbb{R}^n$ is convex).

Convex sets enjoy a lot of interesting properties, this one below is just a small list.

Topology

Exercises

E15.a.4 Let $C \subseteq \mathbb{R}^n$ be a set; show that it is *convex* if and only if it contains every *convex combination* of its points, that is: for every $k \geq 1$, for every choice of $x_1, \dots x_k \in C$, for each choice $t_1, \dots t_k \geq 0$ with $t_1 + \dots + t_k = 1$, you have [16Y]

$$x_1 t_1 + \dots + x_k t_k \in C \quad .$$

E15.a.5 `Topics:simplex.` [16Z]

Given $x_0, \dots x_k \in \mathbb{R}^n$, let

$$\left\{ \sum_{i=0}^{k} x_i t_i \, : \, \sum_{i=0}^{k} t_i = 1 \forall i, t_i \geq 0 \right\} \tag{15.a.6}$$

the set of all possible combinations: prove that this set is convex.

When the vectors $x_1 - x_0, x_2 - x_0 \dots x_k - x_0$ are linearly independent, the set defined above is a *simplex* of dimension k.

Show that, if $n = k$, then the simplex has a non-empty interior, equal to

$$\left\{ \sum_{i=0}^{n} x_i t_i \, : \, \sum_{i=0}^{n} t_i = 1 \forall i, t_i > 0 \right\} \tag{15.a.7}$$

E15.a.8 Let $A \subset \mathbb{R}^n$ be a convex set containing at least two points, and V the smallest affine space that contains A (show that this concept is well defined); and, viewing A as a subset of V, prove that A has non-empty inner part. *Hidden solution:* *[UNACCESSIBLE UUID '171']* [170]

E15.a.9 If $A \subset \mathbb{R}^n$ is convex, $x \in A^\circ$ and $y \in A$ then the segment that links them is contained in A°, except possibly for the extreme y, that is, $\forall t \in (0,1), tx+(1-t)y \in A^\circ$. *Hidden solution:* *[UNACCESSIBLE UUID '173']* [172]

E15.a.10 Prerequisites:15.a.9,10.b.5.If $A \subset \mathbb{R}^n$ is convex, $x \in A^\circ$ and $z \in \partial A$ then the segment that connects them is contained in A°, except possibly for the extreme z (i.e. $\forall t \in (0,1), tx + (1-t)z \in A^\circ$). *Hidden solution:* *[UNACCESSIBLE UUID '175']* [174]

E15.a.11 Given $A \subset \mathbb{R}^n$ convex, show that its interior, as well as its closure, are still convex. *Hidden solution:* *[UNACCESSIBLE UUID '177']* [176]

E15.a.12 Prerequisites:10.b.29,15.a.10.Given $A \subset \mathbb{R}^n$ convex with non-empty interior, show that $\overline{A} = \overline{(A^\circ)}$ (the closure of the interior of A). Then find a simple example of A for which $\overline{A} \neq \overline{(A^\circ)}$.*Hidden solution:* *[UNACCESSIBLE UUID '179']* [178]

E15.a.13 Prerequisites:8.13.Difficulty:*. [17B]

Given $A \subset \mathbb{R}^n$ convex, show that $A^\circ = \left(\overline{A}\right)^\circ$ (the inner part of the closure of A).

Using 15.a.18 it is easily shown that $A^\circ \supseteq \left(\overline{A}\right)^\circ$; unfortunately this result is useful in one of the possible proofs of 15.a.18 (!); an alternative proof uses simplexes as neighbourhoods, cf *15.a.5. Hidden solution:* *[UNACCESSIBLE UUID '17C']*

E15.a.14 Suppose that $C_i \subseteq \mathbb{R}^n$ are convex sets, for $i \in I$: prove that [059]

$$\bigcap_{i\in I} C_i$$

is convex.

Definition 15.a.15. *Let then $A \subseteq \mathbb{R}^n$ be a non-empty set, the **convex hull** (or **convex envelop**) $co(A)$ of A is the intersection of all convex sets containing A. Because of 15.a.14, $co(A)$ is the smallest convex sets containing A.* [2G4]

See also exercises 12.f.7, 12.g.15 and 12.g.16.

§15.a.a Projection, separation

Exercises

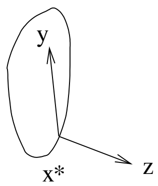
E15.a.16 Topics:projection.Difficulty:*. Note:*This is the well-known "projection theorem", which holds for A convex closed in a Hilbert space; if $A \subset \mathbb{R}^n$ then the proof is simpler, and it's a useful exercise..* [17D]

Given $A \subset \mathbb{R}^n$ closed convex non-empty and $z \in \mathbb{R}^n$, show that there is only one minimum point x^* for the problem

$$\min_{x\in A} \|z - x\| \ .$$

Show that x^* is the minimum if and only if

$$\forall y \in A, \langle z - x^*, y - x^* \rangle \le 0 \ .$$



x^* is called "the projection of z on A".

(Note that this last condition is simply saying that the angle must be obtuse.)

Hidden solution: [UNACCESSIBLE UUID '17G']

E15.a.17 Topics:separation. Prerequisites:15.a.16. [17H]

Given $A \subset \mathbb{R}^n$ closed non-empty convex and $z \notin A$, let x^* be defined as in the previous exercise 15.a.16; define $\delta = \|z - x^*\|$, $v = (z - x^*)/\delta$ and $a = \langle v, x^* \rangle$. Prove that v, a and $v, a + \delta$ define two parallel hyperplanes that strongly separate z from A, in the sense that $\langle z, v \rangle = a + \delta$ but $\forall x \in A, \langle x, v \rangle \le a$.

E15.a.18 Topics:separation.Difficulty:*. [17J]

This result applies in very general contexts, and is a consequence of Hahn–Banach theorem (which makes use of Zorn's Lemma); if $A \subset \mathbb{R}^n$ it can be proven in an elementary way, I invite you to try.

Given $A \subset \mathbb{R}^n$ open convex non-empty and $z \notin A$, show that there is a hyperplane P separating z from A, that is, $z \in P$ while A is entirely contained in one of the two closed half-spaces bounded by the hyperplane P. Equivalently, in analytical form, there exist $a \in \mathbb{R}, v \in \mathbb{R}^n, v \ne 0$ such that $\langle z, v \rangle = a$ but $\forall x \in A, \langle x, v \rangle < a$; and

$$P = \{ y \in \mathbb{R}^n \, : \, \langle y, v \rangle = a \} \, .$$

The hyperplane P thus defined is called *supporting hyperplane* of z for A.

There are (at least) two possible proofs. A possible proof is made by induction on n; we can assume without loss of generality that $z = e_1 = (1, 0 \ldots 0), 0 \in A, a = 1$; keep in mind that the intersection of a convex open sets with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ is an open convex set in $\mathbb{R}^{n-1}$; this proof is complex but does not use any prerequisite. A second proof uses 15.a.11 and 15.a.17 if $z \notin \partial A$; if $z \in \partial A$ it also uses 15.a.12 to find $(z_n) \subset (A^c)^\circ$ with $z_n \to z$. Hidden solution: [UNACCESSIBLE UUID '17K']

E15.a.19 Prerequisites:15.a.18,12.f.3.If A, B are disjoint convex, with A open, show that there is a hyperplane separating A and B, that is, there exist $v \in \mathbb{R}^n, v \ne 0$ and $c \in \mathbb{R}$ such that [17M]

$$\forall x \in A, \langle x, v \rangle < c \text{ but } \; \forall y \in B, \langle y, v \rangle \ge c \, ; \qquad (15.a.20)$$

moreover show that if also B is open, then you can have strict separation (i.e. strict inequality in the last term in (15.a.20)).

(Hint: given $A, B \subseteq \mathbb{R}^n$ convex nonempty, show that

$$A - B \stackrel{def}{=} \{ x - y, x \in A, y \in B \}$$

is convex; show that if A is open then $A - B$ is open, as in 12.f.3.) Hidden solution: [UNACCESSIBLE UUID '17N']

E15.a.21 Find an example of open convex sets $A, B \subset \mathbb{R}^2$ with $\overline{A}, \overline{B}$ disjoint, and such that there is a single hyperplane separating them (*i.e.* an "unique" choice of v, c that satisfies (15.a.20); "unique", up to multiplying v, c by the same positive constant). *Hidden solution:* [UNACCESSIBLE UUID '17Q'] [17P]

E15.a.22 Prerequisites:15.a.18.If $A \subset \mathbb{R}^n$ is convex, $x \in A^\circ$ and $y \in \partial A$, then the straight line that connects them, continuing over y, stays out of $\overline{A}$ (i.e. $\forall t > 1, ty + (1 - t)x \notin \overline{A}$). *Hidden solution:* [UNACCESSIBLE UUID '17S'] [17R]

E15.a.23 Topics:separation,support. Prerequisites: 15.a.8,15.a.18, 15.a.13. [17T]

Given $A \subset \mathbb{R}^n$ convex non-empty and $z \in \partial A$, prove that there exist $v \in \mathbb{R}^n, a \in \mathbb{R}$ such that $\langle z, v\rangle = a$ and $\forall x \in A, \langle x, v\rangle \le a$. The hyperplane thus defined is called *support hyperplane* of z for A. *Hidden solution:* [UNACCESSIBLE UUID '17V']

E15.a.24 Difficulty:*.Given a set $A \subset \mathbb{R}^2$ bounded convex open nonempty, show that ∂A is support of a closed simple arc (that is also Lipschitz continuous). [17W]

Hidden solution: [UNACCESSIBLE UUID '17X']

§15.b Convex function

Definition 15.b.1. *Let $C \subset \mathbb{R}^n$ be a convex set, and $f : C \to \mathbb{R}$ a function. f is* convex *if* [17Y]

$$\forall t \in [0,1], \quad \forall x, y \in C, \quad f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \ .$$

f is strictly convex *if also*

$$\forall t \in (0,1), \quad \forall x, y \in C, x \ne y, \quad f\big(tx + (1-t)y\big) < tf(x) + (1-t)f(y) \ .$$

Definition 15.b.2. *f is said (strictly) concave if $-f$ is (strictly) convex.* [17Z]

Convex functions enjoy a lot of interesting properties, this one below is just a small list.

... equivalent definitions

Exercises

E15.b.3 Let $C \subset \mathbb{R}^n$ be a convex set. Let $f : C \to \mathbb{R}$ be convex; let $x_1, \dots, x_n \in C$ and $t_1, \dots, t_n \in [0,1]$ be such that $\sum_{i=1}^n t_i = 1$. Show that [180]

$$\sum_{i=1}^n t_i x_i \in C$$

and

$$f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i f(x_i) \ .$$

E15.b.4 Let $C \subset \mathbb{R}^n$ be a convex set. Let $f : C \to \mathbb{R}$, show that f is convex if and only if the epigraph [181]

$$\{(x, y) \mid x \in C \ , \ f(x) \le y\}$$

is a convex subset of $C \times \mathbb{R}$.

Properties

The following is a list of properties for convex functions $f : C \to \mathbb{R}$ with $C \subseteq \mathbb{R}^n$. Obviously these properties also apply when $n = 1$; but when $n = 1$ proofs are usually easier, see the next section.

Exercises

E15.b.5 Let $C \subseteq \mathbb{R}^n$ be a convex set, and $f : C \to \mathbb{R}$ a convex function. Given $l \in \mathbb{R}$, define the *sublevel set* as [182]

$$L_l = \{x \in \mathbb{R}^n : f(x) \le l\} \quad .$$

Show that L_l is a convex (possibly empty) set. Deduce that the minimum points of f are a convex (possibly empty) set. Show that if f is strictly convex there can be at most one minimum point.

E15.b.6 Let $C \subset \mathbb{R}^n$ be a convex set; suppose that $f_i : C \to \mathbb{R}$ are convex, where $i \in I$ (a non-empty, and arbitrary, family of indices), and we define $f(x) = \sup_{i \in I} f_i(x)$, where we suppose (for simplicity) that $f(x) < \infty$ for every i: show that f is convex. [183]

E15.b.7 Prerequisites:15.b.4,15.a.23.Difficulty:*.Let $C \subseteq \mathbb{R}^n$ be a convex set, let $f : C \to \mathbb{R}$ be a convex function, we fix $z \in C^\circ$: show that there exists $v \in \mathbb{R}^n$ such that [184]

$$\forall x \in C, f(x) \ge f(z) + \langle v, x - z \rangle \quad . \tag{15.b.8}$$

The plane thus defined is called *support plan* for f in z. Note:*It is preferable not to assume that f is continuous, while proving this result, as this result is generally used to prove that f is continuous!*.*Hidden solution:* *[UNACCESSIBLE UUID '185']*

E15.b.9 Prerequisites:12.a.1,12.10,15.b.7.Difficulty:*. [186]

Let $C \subset \mathbb{R}^n$ be an open convex set, and $f : C \to \mathbb{R}$ a convex function, show that f is continuous.

Note:*In the case of dimension $n = 1$, the proof is much easier, see 15.c.5.*

Hidden solution: *[UNACCESSIBLE UUID '187']*

E15.b.10 Topics:subdifferential.Prerequisites:15.b.7.Difficulty:*. [188]

Let $C \subseteq \mathbb{R}^n$ be an open convex set, and $f : C \to \mathbb{R}$ a convex function; Given $z \in C$, we define the *subdifferential* $\partial f(z)$ as the set of v for which the relation (15.b.8) is valid (i.e., $\partial f(z)$ contains all vectors v defining the support planes to f in z).

$\partial f(z)$ enjoys interesting properties.

- $\partial f(z)$ is *locally bounded*: if $z \in C$ and $r > 0$ is such that $B(z, 2r) \subset C$, then $L > 0$ exists such that $\forall y \in B(z, r), \forall v \in \partial f(x)$ you have $|v| \le L$. In particular, for every $z \in C$, we have that $\partial f(z)$ is a bounded set.
- Show that ∂f is *upper continuous* in this sense: if $z \in C$ and $(z_n)_n \subset C$ and $v_n \in \partial f(z_n)$ and if $z_n \to_n z$ and $v_n \to_n v$ then $v \in \partial f(z)$. In particular, for every $z \in C$, $\partial f(z)$ is a closed set.

Hidden solution: *[UNACCESSIBLE UUID '189']*

E15.b.11 Topics:minimum. Prerequisites:15.b.10.Let $C \subseteq \mathbb{R}^n$ be a convex set, and $f : C \to \mathbb{R}$ a convex function. Show that $z \in C^\circ$ is a minimum if and only if $0 \in \partial f(z)$. [18B]

E15.b.12 Prerequisites:15.b.7,15.b.10.Note:*A vice versa of 15.b.6.* [18C]

Let $C \subset \mathbb{R}^n$ be an open convex set; suppose that $f : C \to \mathbb{R}$ is convex; sequences $(a_h)_h \subseteq \mathbb{R}, (v_h)_h \in \mathbb{R}^n$ exist (for $h \in \mathbb{N}$), such that $f(x) = \sup_{h \in \mathbb{N}}(a_h + v_h \cdot x)$. *Hidden solution:* *[UNACCESSIBLE UUID '18D']*

§15.c Real case

Let $I \subset \mathbb{R}$, then I is convex if and only it is an interval (see 10.f.1). In the following we will consider $f : I \to \mathbb{R}$ where $I = (a, b)$ is an open interval.

Exercises

E15.c.1 Show that $f(x)$ is convex if and only if the map $R(x, y) = \frac{f(x)-f(y)}{x-y}$ is monotonically weakly increasing in x. [†95] Moreover, f is strictly convex if and only if R is strictly increasing. *Hidden solution:* *[UNACCESSIBLE UUID '18G']* [18F]

E15.c.2 Show that for a convex function $f : (a, b) \to \mathbb{R}$ there are only three possibilities: [18H]

- f is strictly increasing
- f is strictly decreasing
- There are two values $l_- \le l_+$ such that f is strictly increasing in $[l_+, b)$, f is strictly decreasing in $(a, l_-]$, and the interval $[l_-, l_+]$ are all minimum points of f;

If also f is strictly convex then there is at most only one minimum point.

E15.c.3 Let $f : (a, b) \to \mathbb{R}$ be convex. Show that, for every closed interval $I \subset (a, b)$, there exists a constant C such that $f|_I$ is Lipschitz with constant C. Provide an example of a continuous and convex function defined on a closed interval that is not Lipschitz. [18J]

E15.c.4 Prove that a continuous function $f : (a, b) \to \mathbb{R}$ is convex if and only if, for every $u, v \in (a, b)$, [18K]

$$f\left(\frac{u+v}{2}\right) \le \frac{f(u)+f(v)}{2} \quad .$$

§15.c.a Convexity and derivatives

Exercises

E15.c.5 Prerequisites:15.c.1.Let $f : (a, b) \to \mathbb{R}$ be convex. [18M]

1. Show that, at every point, right derivative $d^+(x)$ and left derivative $d^-(x)$ exist (In particular f is continuous).
2. Show that $d^-(x) \le d^+(x)$,
3. while, for $x < y$, $d^+(x) \le R(x, y) \le d^-(y)$.
4. hence $d^+(x)$ and $d^-(x)$ are monotonic weakly increasing.
5. Show that $d^+(x)$ is right continuous, while $d^-(x)$ is left continuous.
6. Also show that $\lim_{s\to x-} d^+(s) = d^-(x)$, while $\lim_{s\to x+} d^-(s) = d^+(x)$. In particular d^+ is continuous in x, if and only if d^- is continuous in x, if and only if $d^-(x) = d^+(x)$.
   So d^+, d^- are, so to speak, the same monotonic function, with the exception that, at any point of discontinuity, d^+ assumes the value of the right limit while d^- the value of the left limit.

[†95] Note that $R(x, y)$ is symmetrical.

7. Use the above to show that f is differentiable in x if and only if d^+ is continuous in x, if and only if d^- is continuous in x.
8. Eventually, prove that f is differentiable, except in a countable number of points.

Hidden solution: *[UNACCESSIBLE UUID '18N']*

E15.c.6 Prerequisites:15.c.1.If $f : (a,b) \to \mathbb{R}$ is differentiable, then f is convex if and only if f' is weakly increasing. *Hidden solution:* *[UNACCESSIBLE UUID '18Q']* [18P]

E15.c.7 Prerequisites:15.c.1,15.c.6.If $f : (a,b) \to \mathbb{R}$ is differentiable, then f is strictly convex, if and only if f' is strictly increasing. *Hidden solution:* *[UNACCESSIBLE UUID '18S']* [18R]

E15.c.8 Prerequisites:15.c.1, 15.c.6.Suppose that $f : (a,b) \to \mathbb{R}$ is twice differentiable. f is convex if and only if $f'' \ge 0$ at every point. *Hidden solution:* *[UNACCESSIBLE UUID '18V']* [18T]

E15.c.9 Prerequisites:15.c.8. [18W]

Let $J \subset \mathbb{R}$ be an open nonempty interval, and $f : J \to \mathbb{R}$ be a twice differentiable and convex function. Show that the following facts are equivalent:

1. f is strictly convex,
2. the set $\{x \in J : f''(x) = 0\}$ has an empty interior,
3. f' is monotonic strictly increasing.

Hidden solution: *[UNACCESSIBLE UUID '18X']*

See also the exercise 16.13 for the relationship between integral and convexity.

§15.c.b Convex functions with extended values

We consider convex functions that can also take on value $+\infty$. Let I be an interval.

Exercises

E15.c.10 Let $f : I \to \mathbb{R} \cup \{\infty\}$ be convex, show that $J = \{x \in I : f(x) < \infty\}$ is an interval. [18Y]

E15.c.11 Note:*another vice versa of 15.b.6.* [18Z]

Given $I \subseteq \mathbb{R}$ interval and $f : I \to \mathbb{R} \cup \{\infty\}$ convex and lower semicontinuous, there exist sequences $a_n, b_n \in \mathbb{R}$ such that $f(x) = \sup_n (a_n + b_n x)$.

Hidden solution: *[UNACCESSIBLE UUID '190']*

§15.d Additional properties and exercises

Exercises

E15.d.1 Let $C \subset \mathbb{R}^n$ be a convex set, $f : C \to \mathbb{R}$ a convex function, and $g : \mathbb{R} \to \mathbb{R}$ a convex and weakly increasing function: prove that $f \circ g$ is convex. [191]

E15.d.2 Let $f : [0,\infty) \to \mathbb{R}$ be concave, with $f(0) = 0$ and f continuous in zero. [192]

- Prove that f is *subadditive*, i.e.

$$f(t) + f(s) \geq f(t + s)$$

for every $t, s \geq 0$. If moreover f is strictly concave and $t > 0$ then

$$f(t) + f(s) > f(t + s) .$$

- Prove that, if $\forall x, f(x) \geq 0$, then f is weakly increasing.
- The other way around? Find an example of $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, continuous, monotonic increasing and subadditive, but not concave.

Hidden solution: [UNACCESSIBLE UUID '193']

E15.d.3 Prove Young inequality: given $a, b > 0$ and $p, q > 1$ such that $1/p + 1/q = 1$ then [194]

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{15.d.4}$$

with equality if and only if $a^p = b^q$; prove this using concavity of the logarithm.

See also 24.16. *Hidden solution:* [UNACCESSIBLE UUID '195']

E15.d.5 Let $\alpha \in (0, 1)$, show that x^α is α-Hölder (possibly using the above results). [196]
Hidden solution: [UNACCESSIBLE UUID '197']

See also exercise 16.29.

§15.d.a Distance function

Exercises

E15.d.6 Topics:Distance function, convex sets. Prerequisites:10.d.3,15.d.8.Let $A \subset \mathbb{R}^n$ be a closed nonempty set, and d_A the *distance function* defined in the exercise 10.d.3, that is $d_A(x) = \inf_{y \in A} |x - y|$. Prove that A is a convex set, if and only if d_A is a convex function. [198]

Hidden solution: [UNACCESSIBLE UUID '199']

E15.d.7 Topics:Distance function, convex sets. Prerequisites:10.d.3,15.a.16. [19B]

Given $A \subset \mathbb{R}^n$ a closed convex set, we define the *distance function* $d_A(x)$ as in 10.d.3; let $z \notin A$ and x^* the projection of z on A (i.e. the point of minimum distance in the definition of $d_A(z)$). Having fixed $v = (z - x^*)/|z - x^*|$, show that $v \in \partial f(z)$; where ∂f is the *subdifferential* defined in 15.b.10.

§15.d.b Strictly convex functions and sets

Exercises

E15.d.8 Let $C \subset \mathbb{R}^n$ be a convex, $f : C \to \mathbb{R}$ a convex function, and $r \in \mathbb{R}$: then $\{x \in C, f(x) < r\}$ and $\{x \in C, f(x) \leq r\}$ are convex (possibly empty) sets. [19C]

Remark 15.d.9. *The vice versa is also true: given $A \subset \mathbb{R}^n$ a closed convex set, a convex function $f : \mathbb{R}^n \to \mathbb{R}$ such that $A = \{x : f(x) \leq 0\}$ always exists: For example, you can use $f = d_A$, as seen in 15.d.7 in the previous section.* [23N]

One wonders now, what if f is strictly convex?

Definition 15.d.10. *A closed convex set $A \subset \mathbb{R}^n$ is said* ***strictly convex*** *if, for every $x, y \in A$ with $x \neq y$ and every $t \in (0,1)$ you have* [19D]

$$(tx + (1-t)y) \in A^\circ \quad .$$

(Note that a strictly convex set necessarily has a non-empty interior).

Remark 15.d.11. *From the exercises 15.a.9 and 15.a.10 it follows that if $x \in A^\circ$ or $y \in A^\circ$ then $(tx + (1-t)y) \in A^\circ$: so the definition is "interesting" when $x, y \in \partial A$.* [19F]

Exercises

E15.d.12 Prerequisites:15.b.9.Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function and $r \in \mathbb{R}$ then $A = \{x, f(x) \le r\}$ is a closed and strictly convex (possibly empty) set. *Hidden solution:* *[UNACCESSIBLE UUID '19H']* [19G]

§16 Riemann integral [19K]

All definitions and theorems needed to solve the following exercises may be found in Chap. 1 in [4], or Chap. 6 of [22].

Exercises

E16.1 Let p be a polynomial (with complex coefficients); fix $\theta \in \mathbb{C}, \theta \neq 0$. Define $f(x) = -\int_0^x e^{-\theta t} p(t)\,\mathrm{d}t$. Show that $f(x) = e^{-\theta x} q(x) - q(0)$ where q is a polynomial that has the same degree as p. Determine the linear map (i.e. the matrix) that transforms the coefficients of p into the coefficients of q; and its inverse. [19M]

Hidden solution: *[UNACCESSIBLE UUID '19N'][UNACCESSIBLE UUID '19P']*

E16.2 Note:*Similar to point 8 from exercise 18.8.*Suppose $f_n : [a,b] \to \mathbb{R}$ are Riemann-integrable, and $f : [a,b] \to \mathbb{R}$ a generic function. [19Q]

Find an example where $f_n \to_n f$ pointwise, f is bounded, but f is not Riemann integrable.

Show that, if the convergence is uniform, then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n \,\mathrm{d}x = \int_a^b f \,\mathrm{d}x \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '19R']*

E16.3 Prerequisites:16.2, 18.4. [19S]

Let $I \subset \mathbb{R}$ be an interval with extremes a, b. Let $f, f_n : I \to \mathbb{R}$ be continuous non-negative functions such that $f_n(x) \nearrow_n f$ pointwise (i.e. for every x and n we have $0 \le f_n(x) \le f_{n+1}(x)$ and $\lim_n f_n(x) = f(x)$). Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \ .$$

(Note if the interval is open or semiopen or unbounded then the Riemann integrals are understood in a generalized sense; in this case the right term can also be $+\infty$).
Hidden solution: *[UNACCESSIBLE UUID '19T']*

The previous result is called *Monotonic Convergence Theorem* and holds in very general hypotheses; in the case of Riemann integrals, however, it can be seen as a consequence of the results 16.2 and 18.4.

E16.4 Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, prove that $g \circ f$ is Riemann integrable. [19V]

Hidden solution: *[UNACCESSIBLE UUID '19W']*

E16.5 Say which of these functions $f : [0,1] \to \mathbb{R}$ are Riemann integrable: [19Y]

1. the characteristic function of the Cantor set.
2. $f(0) = 0$, $f(x) = \sin(1/x)$
3. $f(0) = 0$ and

$$f(x) = \frac{1 - \cos(x)}{x^2 + |\sin(1/x)|}$$

4. $f(x) = 0$ if x is irrational, $f(x) = \cos(1/m)$ if $x = n/m$ with n, m coprime.
5. $f(x) = 0$ if x is irrational, $f(x) = \sin(1/m)$ if $x = n/m$ with n, m coprime.

E16.6 `Prerequisites:Fundamental theorem of integral calculus.` [1B0]

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ has class C^1: prove that

$$\int_a^b f(g(t))g'(t)\,\mathrm{d}t = \int_{g(a)}^{g(b)} f(s)\,\mathrm{d}s \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '1B2']* Note that for this result it is not necessary to assume that g is monotonic.

E16.7 `Prerequisites:regulated functions Sec. §13.b.` [1B3]

Show that a regulated function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

E16.8 `Prerequisites:Regulated functions Sec. §13.b.` [1B4]

Find a Riemann integrable function $f : [0, 1] \to \mathbb{R}$ that is not regulated.

Hidden solution: *[UNACCESSIBLE UUID '1B5']*

E16.9 `Difficulty:*.`Can there be a Riemann integrable function $f : [0, 1] \to \mathbb{R}$ that is not regulated (i.e., it does not allow right and left limits) at any point? *Hidden solution:* *[UNACCESSIBLE UUID '1B7']* [1B6]

E16.10 If $f, g : [A, B] \to \mathbb{R}$ are Riemann integrable, then $h(x) = \max\{f(x), g(x)\}$ is Riemann integrable. [1B8]

E16.11 Find a lower semicontinuous function $f : [0, 1] \to \mathbb{R}$, bounded, but not Riemann integrable. [1B9]

E16.12 We define the Beta function as [1BC]

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\ .$$

1. Show that the integral exists (finite) if and only if $x, y > 0$.
2. Note that $B(x, y) = B(y, x)$
3. Relate $B(n, m)$ to $B(n-1, m+1)$. Then calculate the value of $B(n, m)$ for n, m natural positives.
4. Use the result to calculate

$$\int_0^{\pi/2} \sin(t)^9 \cos(t)^7\,dt\ .$$

Hidden solution: *[UNACCESSIBLE UUID '1BD']*

E16.13 `Prerequisites:convex functions.`Let $I \subset \mathbb{R}$ be an open interval, and $x_0 \in I$. Prove that these two facts are equivalent: [1BF]

1. $F : I \to \mathbb{R}$ is convex.
2. There exists $f : I \to \mathbb{R}$ monotonic (weakly) increasing, and such that $F(x) = F(x_0) + \int_{x_0}^x f(s)\,\mathrm{d}s$,

and verify that you can choose f be the right (or left) derivative of F.

E16.14 Exhibit an integrable function $f : [0,1] \to \mathbb{R}$ such that the derivative of the function $F(x) = \int_a^x f(t)dt$ is not f. *Hidden solution:* *[UNACCESSIBLE UUID '1BH']* [1BG]

E16.15 Calculate explicitly [†96] primitive formulas for [1BJ]

$$\frac{1}{\sin(x)^2} \quad , \quad \frac{1}{\sqrt{1+x^2}} \quad , \quad \frac{1}{2+\sin(x)} \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '1BK']*

E16.16 We define the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ as [1BM]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t \,.$$

- Show that $\Gamma(x)$ is well defined for $x > 0$ real.
- Show that $\Gamma(x+1) = x\Gamma(x)$ and deduce that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.
- Show that $\Gamma(x)$ is analytic.
  (You can assume that derivatives of Γ are $\Gamma^{(n)}(x) = \int_0^\infty (\log t)^n t^{x-1} e^{-t}\, \mathrm{d}t$; those are obtained by derivation under integral sign.)

E16.17 Calculate [1BN]

$$\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$$

seeing it as an approximate sum of a Riemann integral.

E16.18 Prerequisites:17.c.3.Let $a \in \mathbb{R}$, let I be open interval with $a \in I$, and $\varphi_0 : I \to \mathbb{R}$ continuous. [1BP]

We recursively define $\varphi_n : I \to \mathbb{R}$ for $n \geq 1$ via $\varphi_n(x) = \int_a^x \varphi_{n-1}(t)\, \mathrm{d}t$; show that

$$\varphi_{n+1}(x) = \frac{1}{n!} \int_a^x (x-t)^n \varphi_0(t)\, \mathrm{d}t \qquad (16.19)$$

Hidden solution: *[UNACCESSIBLE UUID '1BQ']*

E16.20 Prerequisites:16.18.Note:*See also Apostol [3].* [1BR]

Fix $a \in \mathbb{R}$, and I open interval with $a \in I$; assuming that $f : I \to \mathbb{R}$ is if class C^{n+1}, prove **Taylor's formula with integral remainder**

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, \mathrm{d}t \quad .$$

Hidden solution: *[UNACCESSIBLE UUID '1BS']*

E16.21 Prerequisites:15.c.1, 16.13.Let $I \subset \mathbb{R}$ be an open interval. Suppose that $g : I \to \mathbb{R}$ in Riemann integrable on any bounded closed interval contained in I. Fixed $x, y \in \mathbb{R}$ with $x \neq y$, let [1BT]

$$R(x,y) = \frac{1}{y-x}\int_x^y g(s)\,\mathrm{d}s$$

(with the usual convention that $\int_x^y g(s)\,\mathrm{d}s = -\int_y^x g(s)\,\mathrm{d}s$, so that $R(x,y) = R(y,x)$). If g is monotonic, show that $R(x,y)$ is monotonic in each variable. If g is continuous and $R(x,y)$ is monotonic in each variable, show that g is monotonic.

Hidden solution: [UNACCESSIBLE UUID '1BV']

E16.22 Let $f : [a,b] \to \mathbb{R}$ continuous and such that [1BW]

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0$$

for any $g : [a,b] \to \mathbb{R}$ continuous: prove that $f \equiv 0$.

E16.23 Let's go back to the exercise 7.c.33: computing the Cauchy product of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself, produces the series $\sum_n (-1)^n c_n$ with $c_n = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}$; show that $c_n \to \pi$. [1BX]

Hidden solution: [UNACCESSIBLE UUID '1BY']

E16.24 Difficulty:*.Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, show that [1BZ]

$$\lim_{y\to 0+} \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{x^2+y^2}\,\mathrm{d}x = f(0)$$

(Hint. start with the case when f is constant.)

E16.25 Let $n, m \geq 1$ be integers, and set [1C0]

$$I_{n,m} = \int_0^1 x^n (\log x)^m \,\mathrm{d}x \quad :$$

relate $I_{n,m}$ with $I_{n,m-1}$; use that relation to explicitly calculate

$$\int_0^1 x^n (\log x)^n \,\mathrm{d}x \,.$$

Hidden solution: [UNACCESSIBLE UUID '1C1']

E16.26 Prerequisites:16.25.Difficulty:**.Show identities [1C2]

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \qquad (=\sim 1.291285997\dots) \tag{16.27}$$

$$\int_0^1 x^{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n} \qquad (=\sim 0.783430510712\dots) \tag{16.28}$$

(Hint: use the Taylor series e^z, and substitute $z = x\log(x)$; use the exercise 16.25 above.)

E16.29 Difficulty:*.Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable and $\varphi : \mathbb{R} \to \mathbb{R}$ convex: show that [1C3]

$$\varphi\left(\int_0^1 f(x)) \, \mathrm{d}x\right) \le \int_0^1 \varphi(f(x)) \, \mathrm{d}x \quad . \tag{16.30}$$

This result is known as **Jensen's inequality**.

E16.31 Difficulty:*.Suppose that $f : (0,1) \to (0,\infty)$ is continuous and decreasing and $\int_0^1 f(t) \, \mathrm{d}t < \infty$ then $\lim_{r\to 0} r f(r) = 0$. [1C4]

Other exercises regarding Riemann integration can be found in 14.a.8, 17.c.2, 18.8 (part 8).

†96 Taken from the book by Giaquinta and Modica [8], p. 162 and following.

§17 Differentiable functions [1C5]

Definition 17.1. *Let in the following $A \subseteq \mathbb{R}$ be an open set.* [2D0]

By saying that $f : A \to \mathbb{R}$ is differentiable *we mean* differentiable at any point.

Recall that, given $k \geq 1$ integer, f **is of class** *C^k if f is differentiable k-times and the k-th derivative $f^{(k)}$ is continuous; and f* **is of class** *C^∞ if f is differentiable infinitely many times. (Sometimes we may write $f \in C^k$ to signify that f is of class C^k.)*

To address the following exercises, it may be necessary to know some fundamental results in Analysis and Differential Calculus that may be found *e.g.* in [22, 4]; specifically:

- Lagrange's Theorem [†97] : Theorem 5.10 in in [22], or [61].
- De l'Hôpital' rule, and corollaries: : Theorem 5.13 in in [22], Sec. 7.12 in [4] or [24, 59];
- Taylor's Theorem, and the possible remainders: Theorem 5.15 in in [22], Chap. 7 in [4], or [66].

Exercises

E17.2 Let $I \subseteq \mathbb{R}$ be an open interval. Let $f : I \to \mathbb{R}$ be differentiable, and $x, y \in I$ with $x < y$. Show that if $f'(x) \cdot f'(y) < 0$ then $\xi \in I$ exists with $x < \xi < y$ such that $f'(\xi) = 0$. *Hidden solution:* *[UNACCESSIBLE UUID '1C7']* [1C6]

E17.3 Prerequisites:17.2.Note:*Darboux properties.* [1C8]

Let $A \subseteq \mathbb{R}$ be an open set, and suppose that $f : A \to \mathbb{R}$ is differentiable. We want to show that, for each interval $I \subset A$, the image $f'(I)$ is an interval.

So prove this result. For $x, y \in I$ with $x < y$, let's define $a = f'(x), b = f'(y)$. Let's assume for simplicity that $a < b$. For any c with $a < c < b$, there exists $\xi \in I$ with $x < \xi < y$ such that $f'(\xi) = c$.

(Finally, show that this property actually implies that the image $f'(I)$ of an interval I is an interval.)

Hidden solution: *[UNACCESSIBLE UUID '1C9']*

E17.4 Prerequisites:17.3. [1CB]

Let $I \subseteq \mathbb{R}$ be an open interval. Let $f : I \to \mathbb{R}$ be a differentiable function such that $f'(t) \neq 0$ for every $t \in I$: show then that $f'(t)$ has always the same sign.

Hidden solution: *[UNACCESSIBLE UUID '1CC']*

E17.5 Prerequisites:17.3.Difficulty:*. [1CD]

Find a bounded function $f : \mathbb{R} \to \mathbb{R}$ that maps intervals into intervals, but such that there does not exist $g : \mathbb{R} \to \mathbb{R}$ differentiable at every point and with $f = g'$.

(Note that f cannot be continuous, due to the Fundamental Theorem of Calculus.)

Hidden solution: *[UNACCESSIBLE UUID '1CF']*

E17.6 Suppose that $f : \mathbb{R} \to \mathbb{R}$ be differentiable, with $f' = f$: prove, in an elementary way, that that there exists $\lambda \in \mathbb{R}$ s.t. $f(x) = \lambda e^x$. *Hidden solution:* *[UNACCESSIBLE UUID '1CH']* [1CG]

[†97] *a.k.a.* Mean Value Theorem

E17.7 Find a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is bounded but not continuous. *Hidden solution:* *[UNACCESSIBLE UUID '1CK']* [1CJ]

E17.8 Find a continuous and differentiable function $f : [-1, 1] \to \mathbb{R}$ [†98] whose derivative is unbounded. *Hidden solution:* *[UNACCESSIBLE UUID '1CN']* [1CM]

E17.9 Difficulty:*. Describe a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable and such that the image of $[0, 1]$ using f' is $f'([0, 1]) = (-1, 1)$. [1CP]

Before looking for the example, ponder on this notions. We remember the Darboux property 17.3: the image $f'(I)$ of an interval I is an interval; but this does not say that the image of $f'([0, 1])$ should be a closed and bounded interval. If, however, we also knew that f' is continuous, what could we say of $f'([0, 1])$? So what do you deduce *a priori* about the sought example?

Hidden solution: *[UNACCESSIBLE UUID '1CQ']*

E17.10 Let $I = (a, b) \subset \mathbb{R}$ be an open interval. Let $f : I \to \mathbb{R}$ be differentiable: show that f' is continuous, if and only if for every x [1CV]

$$f'(x) = \lim_{(s,t)\to(x,x), s\neq t} \frac{f(t) - f(s)}{t - s} .$$

Hidden solution: *[UNACCESSIBLE UUID '1CW']*

E17.11 Let f be differentiable in the interval (a, b), let $x_0 \in (a, b)$ and $x_0 < \alpha_n < \beta_n, \beta_n \to x_0$ for $n \to \infty$. Show that if the sequence $\frac{\beta_n - x_0}{\beta_n - \alpha_n}$ is bounded then [1CX]

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} \to_n f'(x_0)$$

Show by example that this conclusion is false if the given condition is not verified.

E17.12 Suppose that a given function $f : (a, b) \to R$ is differentiable at every point of (a, b) except x_0, and that the limit $\lim_{t\to x_0} f(t)$ exists and is finite. Show that f is also differentiable in x_0 and that $f(x_0) = \lim_{t\to x_0} f(t)$. [1CZ]

E17.13 Prerequisites:10.g.8.Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions that can be differentiated at every point. Show that $\max\{f, g\}$ is differentiable, except on a set that is at most countable. *Hidden solution:* *[UNACCESSIBLE UUID '1D2'][UNACCESSIBLE UUID '1D3']* [1D1]

E17.14 Let $f : (a, b) \to \mathbb{R}$ be differentiable and such that, if $f(t) = 0$, then $f'(t) = 0$. Show that the function $g(t) = |f(t)|$ is differentiable. *Hidden solution:* *[UNACCESSIBLE UUID '1D6']* [1D4]

E17.15 Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ a polynomial with all real roots and coefficients all non-zero. Show that the number of positive roots (counted with multiplicity) is equal to the number of sign changes in the sequence of coefficients of p. *[Hint. Use induction on n, using the fact that between two consecutive roots of p there exists a root of p' .]*This result is known as *Descartes' rule of signs*. [1D7]

E17.16 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable, and $a, b \in \mathbb{R}$ with $a < b$. Show that, if $f'(a) = f'(b)$, then ξ exists with $a < \xi < b$ such that [1D9]

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a} .$$

[†98] In this sense: the derivative $f'(x)$ exists and is finite for every $x \in [-1, 1]$; at the extremes $x = -1, 1$ only the right and left derivatives are calculated.

§17.a Higher derivatives [2D1]

Exercises

E17.a.1 Let I be an open interval and $x_0 \in I$, let $f : I \to \mathbb{R}$ be differentiable in I and such that there exists the second derivative f'' in x_0: then show that the limit exists [1DD]

$$\lim_{t\to 0} \frac{f(x_0+t)+f(x_0-t)-2f(x_0)}{t^2}$$

and that it coincides with $f''(x_0)$.

Find then a simple example of f differentiable in $(-1,1)$ and such that the second derivative f'' in $x_0 = 0$ does not exist, but the previous limit exists.

Hidden solution: *[UNACCESSIBLE UUID '1DF']*

E17.a.2 5 Let $n \geq 1$ be an integer. Let I be an open interval and $x_0 \in I$, let $f, g : I \to \mathbb{R}$ be functions $n-1$ times differentiable in the interval, and whose $(n-1)$-th derivative is differentiable in x_0. [1DG]

Show that the product fg is differentiable $n-1$ times in the interval, and its $(n-1)$-th derivative is differentiable in x_0. Write an explicit formula for the n-th derivative $(fg)^{(n)}$ in x_0 of the product of the two functions, (formula that uses derivatives of only f and only g).

(If you don't find it, look in Wikipedia at the General Leibniz rule [55]) .

Hidden solution: *[UNACCESSIBLE UUID '1DH']*

E17.a.3 Difficulty:∗.Let $n \geq 1$ be an integer. Let I, J be open intervals with $x_0 \in I, y_0 \in J$. Let then be given $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ such that $g(I) \subseteq J$, f, g are $n-1$ times differentiable in their respective intervals, their $(n-1)$-th derivative is differentiable in x_0 (resp. y_0) and finally $g(x_0) = y_0$. [1DJ]

Show that the composite function $f \circ g$ is differentiable $n-1$ times in the interval and its derivative $(n-1)$-th is differentiable in x_0.

Then write an explicit formula for the nth derivative $(f \circ g)^{(n)}$ in x_0 of the composition of the two functions, (formula that uses derivatives of f and g).

(If you can't find it, read the wikipedia page [54]; or, see this presentation: *https://drive.google.com/drive/folders/1746bdJ89ZywciaEqvIMlGZ7kKHWVekhb* *).*

Hidden solution: *[UNACCESSIBLE UUID '1DK']*

E17.a.4 Prerequisites:17.a.3,3.1.1.Show that the function [1DM]

$$\varphi(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{17.a.5}$$

is of class C^∞, and for $x > 0$

$$\varphi^{(n)}(x) = e^{-1/x} \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} \frac{(-1)^{m+n}}{x^{m+n}} \quad ,$$

$$\binom{n-1}{m-1} = \frac{(n-1)!}{(n-m)!(m-1)!} \quad .$$

whereas $\varphi^{(n)}(x) = 0$ for each $n \in \mathbb{N}, x \le 0$.

Proceed similarly to

$$\psi(x) = \begin{cases} e^{-1/|x|} & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0 \end{cases} \tag{17.a.6}$$

again $\psi \in C^\infty$ and $\psi^{(n)}(0) = 0$ for each $n \in \mathbb{N}$; but in this case $\psi(x) = 0 \iff x = 0$. *Hidden solution:* *[UNACCESSIBLE UUID '1DN'][UNACCESSIBLE UUID '1DP'][UNACCESSIBLE UUID '1DQ']*

E17.a.7 Let it be given N positive integer. Find an example of a function C^∞ with $\varphi(x) = 0$ for $x < 0$ while $\varphi^{(n)}(x) > 0$ for $0 \le n \le N$ and $x > 0$. [1DR]

Hidden solution: *[UNACCESSIBLE UUID '1DS']* Note however that it cannot be required that all derivatives be positive, because of exercise 20.2.

E17.a.8 What can you put in place of "???" so that the function [1DT]

$$g(x) = \begin{cases} ??? & \text{if } \ 0 < x < 1\,, \\ 1 & \text{if } \ x \ge 1\,, \\ 0 & \text{if } \ x \le 0\,. \end{cases}$$

is C^∞?

More generally, how can two C^∞ functions be connected, so that the whole function is C^∞? Given $f_0, f_1 \in C^\infty$, show [†99] that there is a function $f \in C^\infty$ that satisfies

$$\begin{array}{ll} f(x) = f_0(x) & \text{if } \ x \le 0 \quad , \\ f(x) = f_1(x) & \text{if } \ x \ge 1 \quad . \end{array}$$

Hidden solution: *[UNACCESSIBLE UUID '1DV']*

E17.a.9 `Difficulty:*`.Let $f_0, f_1 : \mathbb{R} \to \mathbb{R}$, $f_0, f_1 \in C^\infty$ with $f_0', f_1' > 0$ and $f_1(1) > f_0(0)$: then one can interpolate with a function $f \in C^\infty$ that satisfies [1DW]

$$\begin{array}{ll} f(x) = f_0(x) & \text{if } \ x \le 0 \\ f(x) = f_1(x) & \text{if } \ x \ge 1 \end{array}$$

so that the interpolant has $f' > 0$.

What if $f_1(1) = f_0(0)$?

Hidden solution: *[UNACCESSIBLE UUID '1DX']*

E17.a.10 `Prerequisites:17.a.4`. Find an example of function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^\infty$ and such that, setting $A = \{x : f(x) = 0\}$, the point 0 will be the only point of accumulation of A, i.e. $D(A) = \{0\}$. Compare this example with Prop. 6.8.4 in the notes [2]; and with the example 20.7. *Hidden solution:* *[UNACCESSIBLE UUID '1F0']* [1DZ]

E17.a.11 `Difficulty:*.Note:`*Hadamard's lemma*. [1F1]

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^∞, and such that $f(0) = 0$. Define, for $x \neq 0$, $g(x) \stackrel{\text{def}}{=} f(x)/x$. Show that g can be prolonged, assigning an appropriate value to $g(0)$, and that the prolonged function is C^∞. What is the relationship between $g^{(n)}(0)$ and $f^{(n+1)}(0)$?

Hidden solution: *[UNACCESSIBLE UUID '1F2']*

E17.a.12 Prerequisites:17.a.11.Difficulty:*.Let $f : \mathbb{R} \to [0, \infty)$ be a function of class C^∞ such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, and $f''(0) \neq 0$: show that [1F4]

$$g(x) = \begin{cases} \sqrt{f(x)} & se\ x \geq 0 \\ -\sqrt{f(x)} & se\ x < 0 \end{cases}$$

is of class C^∞. *Hidden solution:* *[UNACCESSIBLE UUID '1F5']*

E17.a.13 Difficulty:* . Given $x_0 < x_1 < x_2 < \ldots < x_n$ and given real numbers $a_{i,h}$ (with $i, h = 0, \ldots n$) show that there is a polynomial $p(x)$ such that $p^{(i)}(x_h) = a_{i,h}$. [1F7]

This result is the starting point of the Hermit method of polynomial interpolation, see [57].

Hidden solution: *[UNACCESSIBLE UUID '1F8']*

E17.a.14 Prerequisites:convex functions.Note:*Exercise 1, written exam March 1st, 2010.* [1F9]

Let's consider the functions $f : \mathbb{R} \to \mathbb{R}$ of class C^∞, such that for every fixed $n \geq 0$, $f^{(n)}(x)$ has constant sign (*i.e.* it is never zero) [†100]. We associate to each such function the sequence of signs that are assumed by $f, f', f'' \ldots$.

What are the possible sequences of signs, and what are the impossible sequences?

(E.g. for $f(x) = e^x$, the associated sequence is $+ + + + + \ldots$, which is therefore a possible sequence.)

See also the exercise 20.2.

See also the exercises 15.c.8 and 15.c.9 on the relationship between convexity and properties of derivatives.

§17.b Taylor polynomial [2D2]

Definition 17.b.1 (Landau Symbols)**.** *Let $a \in \overline{\mathbb{R}}$ and I be a neighborhood of a. Let $f, g : I \to \mathbb{R}$. We will say that "$f(x) = o(g(x))$ for x tending to a" if* [†101] [1FB]

$$\forall \varepsilon > 0,\ \exists \delta > 0, x \in I \land\ |x - a| < \delta \Rightarrow |f(x)| \leq \varepsilon |g(x)| \quad .$$

This notation reads like "f is small o of g".
If $g(x) \neq 0$ for $x \neq a$, then equivalently we can write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0 \quad .$$

We will say that "$f(x) = O(g(x))$ for x tending to a" if if there is a constant $c > 0$ and a neighborhood J of a for which $\forall x \in J, |f(x)| \leq c|g(x)|$.
Again, if $g(x) \neq 0$ for $x \neq a$, then equivalently we can write

$$\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} < \infty \quad ,$$

This notation reads like "f is big O of g".
For further information, and more notations, see [50].
This notation is usually attributed to Landau.

[†99] Possibly with a simple construction based on example 17.a.4.
[†100] We agree that $f^{(0)} = f$.
[†101] Consider that $J = \{x \in I : |x - a| < \delta\}$ is a neighborhood of a.

In the following for simplicity we consider only the case in which $\lim_{x\to a} g(x) = 0$; moreover in Taylor's expansion we always have that $g(x) = (x-a)^n$ with $n \geq 1$ integer. †102

Remark 17.b.2. *Attention! The symbols "small o" and "big O" are used differently from other symbols of mathematics. In fact, they can represent different functions, even in the same context! For example, if we write* [1FC]

$$\sin(x) = x + o(x) \quad , \quad \cos(x) = 1 + o(x)$$

the two symbols "$o(x)$" on the right and left represent different functions. Particular care must therefore be taken in showing the properties used in the calculus. When many of such symbols are present, it is advisable to replace them with placeholder function symbols, as in the following examples.

Let's see two examples. Let $a = 0$ for simplicity.

Example 17.b.3. *We informally state this property.* [1FD]

$$\text{If } n \geq m \geq 1 \text{ then } o(x^n) + o(x^m) = o(x^m).$$

To prove it, we convert it into a precise statement. First of all, let's rewrite it like this.

$$\text{If } f(x) = o(x^n) \text{ and } g(x) = o(x^m) \text{ then } f(x) + g(x) = o(x^m).$$

So let's prove it. From the hypotheses,

$$\lim_{x\to 0} f(x)x^{-n} = 0 \textit{ and } \lim_{x\to 0} g(x)x^{-m} = 0$$

then

$$\lim_{x\to 0} \frac{f(x)+g(x)}{x^m} = \lim_{x\to 0} \frac{f(x)}{x^m} + \lim_{x\to 0} \frac{g(x)}{x^m} = \lim_{x\to 0} x^{n-m}\frac{f(x)}{x^n} + 0 = 0.$$

Example 17.b.4. *We informally state this second property* [1FF]

$$\text{If } n \geq 1 \text{ then } o\Big(x^n + o(x^n)\Big) = o(x^n).$$

We rewrite it like this.

$$\text{If } f(x) = o(x^n) \text{ and } g(x) = o(x^n + f(x)) \text{ then } g(x) = o(x^n).$$

We note that, for $x \neq 0$ small, $x^n + f(x)$ is not zero, as there is a neighborhood in which $|f(x)| \leq |x^n/2|$. As a hypothesis we have that $\lim_{x\to 0} f(x)x^{-n} = 0$ and $\lim_{x\to 0} g(x)/(x^n + f(x)) = 0$ then

$$\lim_{x\to 0} \frac{g(x)}{x^n} = \lim_{x\to 0} \frac{g(x)}{x^n + f(x)} \frac{x^n + f(x)}{x^n}$$

but

$$\lim_{x\to 0} \frac{g(x)}{x^n + f(x)} = 0$$

while

$$\lim_{x\to 0} \frac{x^n + f(x)}{x^n} = 1 \quad .$$

†102 Some authors also use the $o(1)$ notation to indicate an infinitesimal quantity for $x \to a$, but this can generate confusion .

Exercises

E17.b.5 Let $a = 0$ for simplicity. Rewrite the following relations, and prove them. [1FG]

- If $n \geq m \geq 1$ then
$$O(x^n)+O(x^m) = O(x^m), \quad o(x^n)+O(x^m) = O(x^m), \quad x^n+O(x^m) = O(x^m) \quad .$$
- If $n > m \geq 1$ then
$$O(x^n) + o(x^m) = o(x^m), \quad x^n + o(x^m) = o(x^m).$$
- For $n, m \geq 1$
$$\begin{aligned} x^n O(x^m) &= O(x^{n+m}) \\ x^n o(x^m) &= o(x^{n+m}) \\ O(x^n)O(x^m) &= O(x^{n+m}) \\ o(x^n)O(x^m) &= o(x^{n+m}) \end{aligned}$$
- $$\int_0^y O(x^n)\, \mathbb{d}x = O(y^{n+1}) \quad \int_0^y o(x^n)\, \mathbb{d}x = o(y^{n+1}) \quad .$$

E17.b.6 Write the Taylor polynomial of $f(x)$ around $x_0 = 0$, using *"Landau's calculus of* $o(x^n)$*"* seen above. [1FJ]

$f(x)$	$=$	$p(x) + o(x^4)$
$(\cos(x))^2$	$=$	$+o(x^4)$
$(\cos(x))^3$	$=$	$+o(x^4)$
$\cos(x)e^x$	$=$	$+o(x^4)$
$\cos(\sin(x))$	$=$	$+o(x^4)$
$\sin(\cos(x))$	$=$	$+o(x^4)$
$\log(\log(e + x))$	$=$	$+o(x^3)$
$(1 + x)^{1/x}$	$=$	$+o(x^3)$

(A little imagination is required to address the last two. To reduce the computations, develop the last two only up to $o(x^3)$*).*

Hidden solution: [UNACCESSIBLE UUID '1FK']

E17.b.7 Find a rational approximation of $\cos(1)$ with error less than $1/(10!) \sim 2.10^{-7}$ [1FM]
Hidden solution: [UNACCESSIBLE UUID '1FN']

E17.b.8 Write Taylor's polynomial of $(1+x)^\alpha$ with $\alpha \in \mathbb{R} \setminus \mathbb{N}$. *(Infer a generalization of the binomial symbol* $\binom{\alpha}{k}$*).* The associated Taylor series is called *binomial series*, it converges for $|x| < 1$. [1FP]

Hidden solution: [UNACCESSIBLE UUID '1FQ']

E17.b.9 Prerequisites:16.20.Note:*From an idea in Apostol's book [3], Chapter 7.3.*Write Taylor's polynomial (around $x_0 = 0$) for $-\log(1 - x)$, integrating [1FR]

$$\frac{1}{(1-x)} = 1 + x + x^2 + \ldots + x^{n-1} + \frac{x^n}{(1-x)} \tag{17.b.10}$$

and compare the "remainder"

$$\int_0^x \frac{t^n}{(1-t)}\,dt \tag{17.b.11}$$

thus obtained with with the "integral remainder" of $f(x) = -\log(1-x)$ (as presented in Exercise 16.20).

Proceed similarly for $\arctan(x)$, integrating

$$1/(1+x^2) = 1 - x^2 + x^4 + \ldots + (-1)^n x^{2n} - (-1)^n x^{2n+2}/(1+x^2) \quad . \tag{17.b.12}$$

Hidden solution: *[UNACCESSIBLE UUID '1FS']*

E17.b.13 `Prerequisites:16.20,17.b.9.Difficulty:.`Evaluate for which $r > 0$ we have that the Taylor remainder of $f(x) = -\log(1-x)$ is infinitesimal in n, uniformly for $|x| < r$; this, using the remainder seen in (17.b.11), using the integral remainder or using the Lagrange remainder. [1FT]

Hidden solution: *[UNACCESSIBLE UUID '1FV']*

See also exercise 16.20.

§17.c Partial and total derivatives, differentials [2D3]

Exercises

E17.c.1 Check that the following partial derivatives exist, and compute them: [1FX]

$$\frac{\partial}{\partial x}\left(4xy + 3x^2y - zy^2\right) \ , \ \frac{\partial}{\partial y}\left(4xy + 3x^2y - zy^2\right)$$

$$\frac{\partial}{\partial x}\frac{ze^{x+|y|}}{1+x^2|y|} \ , \ \frac{\partial}{\partial z}\frac{ze^{x+|y|}}{1+x^2|y|}$$

Hidden solution: *[UNACCESSIBLE UUID '1FY']*

E17.c.2 `Prerequisites:Riemann integral,16.2.`Let $I \subseteq \mathbb{R}$ open interval with $0 \in I$. Given $f = f(x,y) : I \times [0,1] \to \mathbb{R}$ continuous, and such that also $\frac{\partial}{\partial x} f$ exists and is continuous, set [1FZ]

$$g(x) = \int_0^1 f(x,y)\,dy \quad ,$$

show that g is of class C^1, and that

$$g'(x) = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy \, .$$

Hidden solution: *[UNACCESSIBLE UUID '1G0'][UNACCESSIBLE UUID '1G1']*

E17.c.3 `Prerequisites:Riemann integral, 14.a.9, 14.a.8, 17.c.2.`Let [1G2]

$$h(t) = \int_{a(t)}^{b(t)} f(t,z)\,dz$$

where a, b, f are C^1 class functions: show that h is class C^1 and calculate the derivative.

Hidden solution: *[UNACCESSIBLE UUID '1G3']*

E17.c.4 Are the following functions differentiable in $(0,0)$? [1G4]

$$f_1(x,y) = \begin{cases} x+y & \text{if } x>0 \\ x+ye^{-x^2} & \text{if } x\le 0 \end{cases}, \qquad f_2(x,y)=\sqrt{x^2+y^2}$$

$$f_3(x,y) = (\arctan(y+1))^{x+1}, \qquad f_4(x,y)=\max\{x^2,y^2\} \ .$$

Hidden solution: [UNACCESSIBLE UUID '1G5']

E17.c.5 Prerequisites:3.1.1.Let $f : \mathbb{R}^k \to \mathbb{R}$ be of class C^∞. Recall that, by Schwarz's theorem, permutation of the order of partial derivatives does not change the result. Let $N(n,k)$ be the number of partial (potentially different) derivatives of order n: show that $N(n,k) = \binom{n+k-1}{k-1}$ (which is a polynomial with integer coefficients in the variable n, of order $k-1$). *Hidden solution:* [UNACCESSIBLE UUID '1G7'] [1G6]

E17.c.6 Let $W \subseteq \mathbb{R}^n$ be an open nonempty set, fix $\overline{x} \in W$. Let then $\psi : W \to \mathbb{R}$ of class C^2. Let $\nabla\psi(\overline{x})$ be the row vector of coordinates $\frac{\partial}{\partial x_k}\psi(\overline{x})$ (which is the gradient of ψ, a special case of the "Jacobian matrix"); we abbreviate it to $D = \nabla\psi(\overline{x})$ for simplicity; let H be the Hessian matrix of components $H_{h,k} = \frac{\partial^2}{\partial x_k x_h}\psi(\overline{x})$; show the validity of Taylor's formula at the second order [1G8]

$$\psi(\overline{x}+v) = \psi(\overline{x}) + Dv + \frac{1}{2}v^t H v + o(|v|^2)$$

(note that the product Dv is a matrix 1×1 that we identify with a real number, and similarly for $v^t H v$).

E17.c.7 Prerequisites:17.c.6.Let $V, W \subseteq \mathbb{R}^n$ be open nonempty sets, and $G : V \to W$ of class C^2. Fix $\overline{y} \in V$ and $\overline{x} = G(\overline{y}) \in W$. Suppose that $\psi : W \to \mathbb{R}$ is of class C^2; define $\tilde\psi = \psi\circ G$, then compare Taylor's second-order formulas for ψ and $\tilde\psi$ (centered in $\overline{x}$ and $\overline{y}$, respectively). Assuming also that G is a diffeomorphism, verify that [1GB]

- $\overline{x}$ is a stationary point for ψ if and only if $\overline{y}$ is stationary point for $\tilde\psi$,
- and in this case the Hessians of ψ and $\tilde\psi$ are similar (i.e. the matrices are equal, up to coordinate changes).

Hidden solution: [UNACCESSIBLE UUID '1GC']

§17.d Implicit function theorem [2D4]

We will use the Implicit Function Theorem, in the multivariable version (Theorem 7.7.4 in [2]). We recall it here for convenience, with some small changes in notations.

Theorem 17.d.1 (Implicit function theorem in $\mathbb{R}^n$). *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous, with A open, and let $\overline{x} = (\overline{x}', \overline{x}_n) \in A$ be such that $\partial_{x_n} f$ exists in a neighborhood of $\overline{x}$, is continuous in $\overline{x}$ and $\partial_{x_n} f(\overline{x}) \neq 0$. Define $\overline{a} = f(\overline{x})$.* [1GD]

There is then a "cylindrical" neighborhood U of $\overline{x}$

$$U = U' \times J$$

where

$$U' = B(\overline{x}', \alpha)$$

is the open ball in $\mathbb{R}^{n-1}$ *centered in* $\overline{x}'$ *of radius* $\alpha > 0$*, and*

$$J = (\overline{x}_n - \beta, \overline{x}_n + \beta)$$

with $\beta > 0$*. Inside this neighborhood* $U \cap f^{-1}(\{\overline{a}\})$ *coincides with the graph* $x_n = g(x')$*, with* $g : U' \to J$ *continuous.*

This means that, for every $x = (x', x_n) \in U$*,* $f(x) = \overline{a}$ *if and only if* $x_n = g(x')$*.*

Moreover, if f *is of class* C^k *on* A *for some* $k \in \mathbb{N}^*$*, then* g *is of class* C^k *on* U' *and*

$$\frac{\partial g}{\partial x_i}(x') = -\frac{\frac{\partial f}{\partial x_i}(x', g(x'))}{\frac{\partial f}{\partial x_n}(x', g(x'))} \qquad \forall x' \in U', \forall i, 1 \le i \le n-1 \quad . \tag{17.d.2}$$

Exercises

E17.d.3 Consider the following C^∞ function of 2 variables [1GF]

$$f(x, y) = x^3 + y^4 - 1 \quad .$$

Check that $\{f = 0\} = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ is not empty; then, for each point of the plane where f vanishes, discuss whether the implicit function theorem can be applied, and therefore if the set $\{f = 0\}$ is locally graph of a C^∞ function. Also study the set $\{f = 0\}$: is it compact? How many connected components are there?

(Please note what is shown in 17.d.13).

Hidden solution: [UNACCESSIBLE UUID '1GG']

E17.d.4 Repeat the study of the previous exercise 17.d.3 for the function [1GJ]

$$f(x, y) = \sin(x + y) + x^2 \quad .$$

Hidden solution: [UNACCESSIBLE UUID '1GK'][UNACCESSIBLE UUID '1GN']

E17.d.5 Note:*Exercise 2, Written exam, June 30th 2017.*Repeat the study of the previous exercise for the function [1GP]

$$f(x, y) = 1 + 4x + e^x y + y^4 \quad .$$

Show that the zero set is not compact.

E17.d.6 Let $A \subset \mathbb{R}^3$ be an open set and suppose that $f, g : A \to \mathbb{R}$ is differentiable, and such that in $p_0 = (x_0, y_0, z_0) \in A$ we have that $\nabla f(p_0), \nabla g(p_0)$ are linearly independent and $f(p_0) = g(p_0) = 0$: show that the set $E = \{f = 0, g = 0\}$ is a curve in a neighborhood of p_0. [1GQ]

(Hint: consider that the vector product $w = \nabla f(p_0) \times \nabla g(p_0)$ *is nonzero if and only if the vectors are linearly independent — in fact it is formed by the determinants of the minors of the Jacobian matrix. Assuming without loss of generality that* $w_3 \neq 0$*, show that* E *is locally the graph of a function* $(x, y) = \gamma(z)$*.)*

Hidden solution: [UNACCESSIBLE UUID '1GR']

E17.d.7 Note:*Written exam, July 4th 2018.*The figure 5 shows the set $E = \{(x, y) : ye^x + xe^y = 1\}$. [1GS]

Properly prove the following properties:

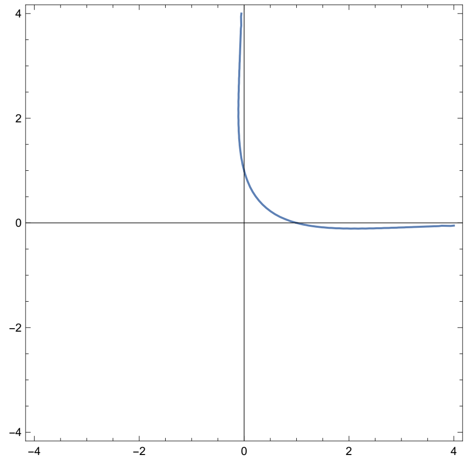


Figure 5: Figure for exercise 17.d.7.

(i) at every point $(x_0, y_0) \in E$ the assumptions of the implicit function theorem are satisfied;

(ii) $E \cap \{(x, y) : x > 0\}$ coincides with the graph, in the form $y = f(x)$, of a single function f defined on $(0, +\infty)$;

(iii) E is connected;

(iv) $\lim_{x\to+\infty} f(x) = 0$.

(v) Show (at least intuitively) that $x_0 > 0$ exists with the property that f is decreasing for $0 < x < x_0$, increasing for $x > x_0$.

Hidden solution: [UNACCESSIBLE UUID '1GV']

E17.d.8 Let E be the set of horizontal lines [1GW]

$$E = \{(x, 0) : x \in \mathbb{R}\} \cup \bigcup_{n=1}^{\infty} \{(x, 1/n) : x \in \mathbb{R}\} \quad .$$

Find a function $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, y)$ class C^1 such that $E = \{(x, y) : f(x, y) = 0\}$.

Prove that necessarily $\partial_y f(0, 0) = 0$.

Set $(\overline{x}, \overline{y}) = (0, 0)$. Note that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $f(x, g(x)) = 0$! In fact, the function $g \equiv 0$ is the only function with such characteristics. Thus part of the thesis in the implicit function theorem is satisfied.

So explain precisely why the thesis of the implicit function theorem is not satisfied.

§17.d.a Extensions

Now let's see some variations of the "standard" theorem.

Exercises

E17.d.9 Prerequisites:14.c.1. [1GX]

We work in the hypotheses of the theorem 17.d.1. Show that, if $f(\cdot, y)$ is Lipschitz of constant L for every fixed y, i.e.

$$|f(x_1', y) - f(x_2', y)| \le L|x_1' - x_2'| \ \ \forall x_1', x_2 \in U', y \in J$$

(and $L > 0$ does not depend on x_1', x_2', y), then g is Lipschitz of constant L'. What is the relationship between the constants L and L'?

Similarly if f is Hölderian.

Hidden solution: [UNACCESSIBLE UUID '1GY']

E17.d.10 In the same assumptions as the previous theorem 17.d.1, show that there exist $\varepsilon > 0$ and a continuous function $\tilde{g} : V \to \mathbb{R}$ where $I = (\overline{a} - \varepsilon, \overline{a} + \varepsilon)$ and $V = U' \times I$ is open in $\mathbb{R}^n$, such that [1GZ]

$$\forall (x', a) \in V \quad , \quad (x', \tilde{g}(x', a)) \in U \quad \text{e} \quad f\big(x', \tilde{g}(x', a)\big) = a \quad . \tag{17.d.11}$$

Vice versa if $x \in U$ and $a = f(x)$ and $a \in I$ then $x_n = \tilde{g}(x', a)$.

Note that the previous relation means that, for each fixed $x' \in U'$, the function $\tilde{g}(x', \cdot)$ is the inverse of the function $f(x', \cdot)$ (when defined on appropriate open intervals).

So, moreover, the function $\tilde{g}$ is always differentiable with respect to a, and the partial derivative is

$$\frac{\partial}{\partial a}\tilde{g}(x', a) = \frac{1}{\frac{\partial}{\partial x_n} f(x', \tilde{g}(x', a))} \quad .$$

The other derivatives instead (obviously) are as in the theorem 17.d.1.

The regularity of $\tilde{g}$ is the same as g: if f is Lipschitz then $\tilde{g}$ is Lipschitz; if $f \in C^k(U)$ then $\tilde{g} \in C^k(V)$.

Hidden solution: [UNACCESSIBLE UUID '1H0']

E17.d.12 In the same hypotheses of the exercise 17.d.10, we also assume that $f \in C^1(A)$. [1H1]

- We decompose $y = (y', y_n), \in \mathbb{R}^n$ as we did for x. We define the function $G : V \to \mathbb{R}^n$ as $G(y) = (y', \tilde{g}(y))$. Let $W = G(V)$ be the image of V, show that $W \subseteq U$ and that W is open.
- Show that is $G : V \to W$ is a diffeomorphism; and that its inverse is the map $F(x) = (x', f(x))$.
- Let's define $\tilde{f} = f \circ G$. Show that $\tilde{f}(x) = x_n$.

(This exercise will be used, together with 17.c.7, to address constrained problems, in Section §17.e). Hidden solution: [UNACCESSIBLE UUID '1H2']

E17.d.13 Prerequisites: 8.e.17, 8.e.18, 10.g.8, 10.g.9, 17.a.8, 17.d.1, 21.7 and 21.a.4. Difficulty:**. [1H3]

For this exercise we need definitions and results presented in the Chapter §21.

Let $r \geq 1$ integer, or $r = \infty$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ of class C^r, and such that $\nabla F \neq 0$ at every point $F = 0$.

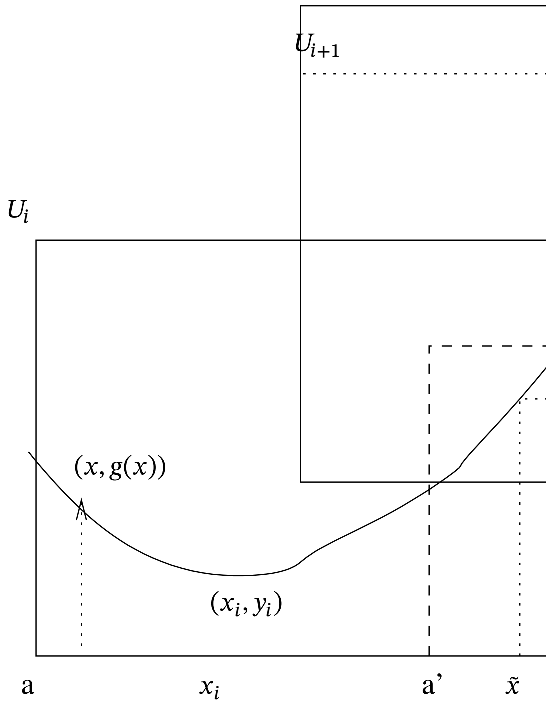
We know, from 8.e.17, that $\{F = 0\}$ is the disjoint union of connected components, and from 8.e.18 that every connected component is a closed.

Show that, for every connected component K, there is an open set $A \supseteq K$ such that $K = A \cap \{F = 0\}$, and that therefore there are at most countably many connected components.

Show that each connected component is the support of a simple immersed curve of class C^r, of one of the following two types:

- the curve is closed, or
- The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is not closed and is unbounded (i.e. $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$).

The first case occurs if and only if the connected component is a compact.



Hidden solution: [UNACCESSIBLE UUID '1H4'][UNACCESSIBLE UUID '1H5']

§17.e Constrained problems [2D5]

Definition 17.e.1. *Let now $A \subseteq \mathbb{R}^n$ an open non-empty set, let $f, \varphi : A \to \mathbb{R}$ be real functions of class C^1 on A. Having fixed $a \in \mathbb{R}$ we then define the level set* [1F6]

$$E_a = \{x \in A : \varphi(x) = a\}$$

we assume that E_a is non-empty, and that $\nabla\varphi(x) \neq 0$ for each $x \in E_a$.

*We call **local minimum point of f bound to** E_a a point of E_a that is a local minimum for $f|_{E_a}$; and similarly for maxima.*

To solve the following exercises it may be useful to apply the results seen in 17.c.6, 17.c.7, 17.d.12.

Exercises

E17.e.2 Prerequisites:17.e.1,17.d.12. Let f, φ be class C^1 in the open set A, and let $\overline{x}$ be a local minimum point for f bound to E_a (so $\varphi(x) = a$). Show that $\lambda \in \mathbb{R}$ exists such that $\nabla f(\overline{x}) + \lambda \nabla \varphi(\overline{x}) = 0$; this λ is called **the Lagrange multiplier**. [1H8]

Hidden solution: [UNACCESSIBLE UUID '1H9']

E17.e.3 Prerequisites:17.e.1,17.c.7,17.e.2.Let f, φ be of class C^2 in the open set A, and let $\overline{x}$ be a minimum point for f constrained to E_a; let λ be the Lagrange multiplier; let's define $h = f(x) + \lambda\varphi(x)$, then [1HB]

$$\forall v, v \cdot \nabla\varphi(x) = 0 \Longrightarrow v \cdot Hv \geq 0$$

where H is the Hessian matrix of h.

Hidden solution: [UNACCESSIBLE UUID '1HC']

E17.e.4 In the same hypotheses, we see a "vice versa". Let $f, \varphi : A \to \mathbb{R}$ be of class C^2 in the open set A, and let $\overline{x} \in E_a$ and $\lambda \in \mathbb{R}$ be such that $\nabla f(\overline{x}) + \lambda\nabla\varphi(\overline{x}) = 0$; suppose that [1HD]

$$\forall v, v \cdot \nabla\varphi(x) = 0 \Longrightarrow v \cdot Hv > 0$$

where

$$h(x) = f(x) + \lambda\varphi(x)$$

and H is the Hessian matrix of h in $\overline{x}$. Show that $\overline{x}$ is a local minimum point for f bound to E_a.

Hidden solution: [UNACCESSIBLE UUID '1HF']

§17.e.a Constraints with inequalities

Now let's consider a different kind of constraint.

Definition 17.e.5. *Let* [2BH]

$$F_a = \{x \in A : \varphi(x) \leq a\} \quad ;$$

we always assume that F_a is non-empty and that $\nabla\varphi(x) \neq 0$ for each $x \in E_a$.

*We call **local minimum point of f bound to** F_a a point of F_a that is of local minimum for $f|_{F_a}$; and similarly for maxima.*

Exercises

E17.e.6 Prerequisites:17.e.5,17.e.1.Show that $\partial F_a = E_a$ and that F_a coincides with the closure of its interior. *(Topological operations must be performed within A, seen as a topological space!)* [1HG]

E17.e.7 Prerequisites:17.e.5.Show that a necessary condition for $x \in A$ to be a local minimum of f bound to F_a, is that, [1HH]

- either $\varphi(x) < a$ and $\nabla f(x) = 0$,
- or $\varphi(x) = a$ and $\nabla f(x) + \lambda\nabla\varphi(x) = 0$ with $\lambda \geq 0$.

These are the **Karush–Kuhn–Tucker** conditions.

Hidden solution: [UNACCESSIBLE UUID '1HJ']

E17.e.8 Prerequisites:17.e.5.In the case $n = 1$, suppose A is an open interval, show that if $\varphi(x) = a$ and $f'(x)\varphi'(x) < 0$ then the point x is a local minimum point for f bound to F_a. [1HK]

E17.e.9 Prerequisites:17.e.5.Find a simple example in the case $n = 2$ where the point x is not a local minimum for f bound to F_a, but $\varphi(x) = a$ and $\nabla f(x) + \lambda\nabla\varphi(x) = 0$ with $\lambda > 0$. [1HM]

Hidden solution: *[UNACCESSIBLE UUID '1HN']*

§18 Limits of functions [1HQ]

Definition 18.1. *Consider a set A, a function $f : A \to \mathbb{R}$ and a sequence of functions $f_n : A \to \mathbb{R}$. We will say that f_n converges to f pointwise if* [2DT]

$$\forall x \in A\,,\ \lim_{n\to\infty} f_n(x) = f(x) \quad .$$

We will say that f_n converges to f uniformly if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N}, \forall n \ge N, \forall x \in A\,,\ |f_n(x) - f(x)| < \varepsilon \quad .$$

Further informations on these subjects may be found in Chap. 6 of [2], Chap. 11 in [4], or Chap. 7 of [22].

Definition 18.2. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $\mathcal{F}$ be a family of functions $f : X_1 \to X_2$, we will say that it is an* ***equicontinuous family*** *if one of these equivalent properties holds.* [1HR]

- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in \mathcal{F}$

$$\forall x, y \in X_1,\ d_1(x, y) \le \delta \Rightarrow d_2(f(x), f(y)) \le \varepsilon \quad .$$

- *There exists a a fixed monotonically weakly increasing function $\omega : [0, \infty) \to [0, \infty]$, for which $\lim_{t\to 0+} \omega(t) = \omega(0) = 0$ (ω is called* "continuity modulus" [†103]*) such that*

$$\forall f \in \mathcal{F},\ \forall x, y \in X_1,\ d_2(f(x), f(y)) \le \omega\big(d_1(x, y)\big) \quad . \tag{18.3}$$

- *There exists a fixed continuous function $\omega : [0, \infty) \to [0, \infty]$ with $\omega(0) = 0$ such that (18.3) holds.*

(The result 14.a.11 can be useful to prove equivalence of the last two clauses.)

Exercises

E18.4 Note:*This result is known as "Dini's lemma".* [1HS]

Let (X, d) be a metric space, let $I \subset X$ be a compact set, and suppose that $f, f_n : I \to \mathbb{R}$ are continuous and such that $f_n(x) \searrow_n f(x)$ pointwise (i.e. for every $x \in I$ and n we have $f(x) \le f_{n+1}(x) \le f_n(x)$ and $\lim_n f_n(x) = f(x)$). Show that $f_n \to f$ uniformly.

Hidden solution: [UNACCESSIBLE UUID '1HT']

Hidden solution: [UNACCESSIBLE UUID '1HV']

In following exercises we will see that, if even one of the hypotheses fails, then there are counterexamples.

E18.5 Find an example of continuous and bounded functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \searrow_n 0$ pointwise, but not $f_n \to 0$ uniformly. [1HW]

Hidden solution: [UNACCESSIBLE UUID '1HX']

E18.6 Find an example of continuous and bounded functions $f_n : [0,1] \to [0,1]$ such that $f_n(x) \to_n 0$ pointwise but not $f_n \to 0$ uniformly. [1HY]

Hidden solution: [UNACCESSIBLE UUID '1HZ']

E18.7 Find an example of functions $f_n : [0,1] \to [0,1]$ continuous, bounded, and such that $f_n(x) \searrow_n f(x)$ pointwise to $f : [0,1] \to [0,1]$ (i.e. for every x and n we have $0 \le f_{n+1}(x) \le f_n(x) \le 1$ and $\lim_n f_n(x) = f(x)$) but f is not continuous and the convergence $f_n \to f$ is not uniform. [1J1]

Hidden solution: [UNACCESSIBLE UUID '1J2']

E18.8 Let $I \subset \mathbb{R}$ be an interval. Which of these classes $\mathcal{F}$ of functions $f : I \to \mathbb{R}$ are closed for uniform convergence? Which are closed for pointwise convergence? [1J3]

1. The continuous and monotonic (weakly) increasing functions on $I = [0,1]$.
   Hidden solution: [UNACCESSIBLE UUID '1J4']
2. The convex functions on $I = [0,1]$.
   Hidden solution: [UNACCESSIBLE UUID '1J5']
3. Given $\omega : [0,\infty) \to [0,\infty)$ a fixed continuous function with $\omega(0) = 0$ (which is called *"continuity modulus"*), and

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R} \ : \ \forall x, y, |f(x) - f(y)| \le \omega(|x-y|)\}$$

   (this is called *a family of equicontinuous functions*, as explained in the definition 18.2.)
   Hidden solution: [UNACCESSIBLE UUID '1J6']
4. Given $N \ge 0$ fixed, the family of all polynomials of degree less than or equal to N, seen as functions $f : [0,1] \to \mathbb{R}$.
   Hidden solution: [UNACCESSIBLE UUID '1J7']
5. The regulated functions on $I = [0,1]$. [†104]
   Hidden solution: [UNACCESSIBLE UUID '1J9']
6. The uniformly continuous and bounded functions on $I = \mathbb{R}$.
   Hidden solution: [UNACCESSIBLE UUID '1JB']
7. The Hoelder functions on $I = [0,1]$, i.e.

$$\left\{f : [0,1] \to \mathbb{R} \,\middle|\, \exists b > 0, \exists \alpha \in (0,1]\ \forall x, y \in [0,1], |f(x) - f(y)| \le b|x-y|^\alpha \right\}$$

   Hidden solution: [UNACCESSIBLE UUID '1JC'][UNACCESSIBLE UUID '1JD']
8. The Riemann integrable functions on $I = [0,1]$.
   Hidden solution: [UNACCESSIBLE UUID '1JF']

E18.9 We wonder if the previous classes $\mathcal{F}$ enjoy a *"rigidity property"*, that is, if from a more "weak" convergence in the class follows a more "strong" convergence. Prove the following propositions. [1JG]

[†103] See also 14.b.2, regarding the notion of "continuity modulus".

[†104] *Regulated* functions $f : I \to \mathbb{R}$ are the functions that, at each point, have finite left limit, and finite right limit. See Section §13.b.

1. Let $f_n, f : I \to \mathbb{R}$ be continuous and monotonic (weakly) increasing functions, defined over a closed and bounded interval $I = [a, b]$. Suppose there is a dense set J in I with $a, b \in J$, such that $\forall x \in J, f_n(x) \to_n f(x)$, then $f_n \to_n f$ uniformly. *Hidden solution:* [UNACCESSIBLE UUID '1JH']
2. Let $A \subseteq \mathbb{R}$ be open interval. Let $f_n, f : A \to \mathbb{R}$ be convex functions on A. If there is a set J dense in A such that $\forall x \in J, f_n(x) \to_n f(x)$, then, for every $[a, b] \subset A$, we have that $f_n \to_n f$ uniformly on $[a, b]$.
   Hidden solution: [UNACCESSIBLE UUID '1JJ']
3. Let $f_n : I \to \mathbb{R}$ be a family of equicontinuous functions, [†105] defined on a closed and bounded interval $I = [a, b]$, and let ω be their modulus of continuity. If there is a set J dense in $[a, b]$ such that $\forall x \in J, f_n(x) \to_n f(x)$, then, f extends from J to I so that it is continuous (with modulus ω), and $f_n \to_n f$ uniformly on $[a, b]$.
   Hidden solution: [UNACCESSIBLE UUID '1JK']
4. Let $f_n, f : I \to \mathbb{R}$ be polynomials of degree less than or equal to N, seen as functions defined on an interval $I = [a, b]$ closed and bounded; fix $N + 1$ distinct points $a \le x_0 < x_1 < x_2 < \ldots < x_N \le b$; assume that, for each x_i, $f_n(x_i) \to_n f(x_i)$: then f_n converge to f uniformly, and so do each of their derivatives $D^k f_n \to_n D^k f$ uniformly.
   Hidden solution: [UNACCESSIBLE UUID '1JM']

Also look for counterexamples for similar propositions, when applied to the other classes of functions seen in the previous exercise.

E18.10 Prerequisites:18.2, 18.8 subpoint 6.Difficulty:*. [1JN]

If $f_n, f : I \to \mathbb{R}$ are uniformly continuous on a set $I \subset \mathbb{R}$, and $f_n \to_n f$ uniformly on I, then f is uniformly continuous, and the family $(f_n)_n$ is equicontinuous.

Hidden solution: [UNACCESSIBLE UUID '1JP']

E18.11 Let $f : \mathbb{R} \to \mathbb{R}$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be the translations of f, defined (for $t \in \mathbb{R}$) by $g_t(x) = f(x - t)$. Show that g_t tends pointwise to f for $t \to 0$, if and only if f is continuous; and that g_t tends uniformly to f for $t \to 0$, if and only if f is uniformly continuous. [1JQ]

Hidden solution: [UNACCESSIBLE UUID '1JR']

E18.12 Let $I \subset \mathbb{R}$ be an open set, and let $\hat{x}$ be an accumulation point for I [†106] , Let $f_m : I \to \mathbb{R}$ be a sequence of bounded functions that converge uniformly to $f : I \to \mathbb{R}$ when $m \to \infty$. Suppose that, for every m, there exists the limit $\lim_{x \to \hat{x}} f_m(x)$, then [1JS]

$$\lim_{m\to\infty} \lim_{x\to\hat{x}} f_m(x) = \lim_{x\to\hat{x}} \lim_{m\to\infty} f_m(x)$$

in the sense that if one of the two limits exists, then the other also exists, and they are equal. (The above result also applies to right limits or left limits.)

Show with a simple example that, if the limit is not uniform, then the previous equality does not hold.

Hidden solution: [UNACCESSIBLE UUID '1JT'] (See also the exercise 7.a.8).

[†105] Definition is in 18.2

[†106] Including also the case where I is not upper bounded, and $\hat{x} = +\infty$; or the case where I is not lower bounded and $\hat{x} = -\infty$.

E18.13 Let $I \subset \mathbb{R}$ be a compact interval, let $f_n, f : I \to \mathbb{R}$ be continuous. Show that the following two facts are equivalent. [1JV]

a. For every $x \in X$ and for every sequence $(x_n)_n \subset I$ for which $x_n \to_n x$, we have $\lim_{n\to\infty} f_n(x_n) = f(x)$;

b. $f_n \to_n f$ uniformly on I.

Then find an example where $I = [0, 1)$, the first point holds, but f_n does not tend uniformly to f.

Hidden solution: [UNACCESSIBLE UUID '1JW']

§18.a On Ascoli–Arzelà's Theorem

Now we'll see some exercises that reconstruct the famous Ascoli–Arzelà Theorem.

Exercises

E18.a.1 Prerequisites:18.8 subpoint 6,18.10. Let $I \subseteq \mathbb{R}$ be a subset. Let X be the set of functions $f : I \to \mathbb{R}$ bounded and uniformly continuous. We equip X with distance $d_\infty(f, g) = \|f - g\|_\infty$. Show that the metric space (X, d_∞) is complete. *Hidden solution:* [UNACCESSIBLE UUID '1JY'] In particular, X is a closed vector subspace of the space $C_b(I)$ of continuous and bounded functions. [1JX]

E18.a.2 Prerequisites:18.2,18.8.6,18.10.Difficulty:**. [1K0]

Define (X, d_∞) as in the previous exercise 18.a.1. Fix now $\mathcal{F} \subseteq X$ a family of functions, suppose $\mathcal{F}$ is totally bounded (as defined in 10.j.1): Show then that the family $\mathcal{F}$ is equicontinuous.

Hidden solution: [UNACCESSIBLE UUID '1K1']

E18.a.3 Prerequisites:18.2,10.j.6, 18.9.3.Difficulty:*. [1K2]

Let now $I \subseteq \mathbb{R}$ be a closed and bounded interval. Let $f_n : I \to \mathbb{R}$ continuous functions, and suppose that the sequence (f_n) is equicontinuous and bounded (i.e. $\sup_n \|f_n\|_\infty < \infty$). Show that there is a subsequence f_{n_k} that converges uniformly.

Hidden solution: [UNACCESSIBLE UUID '1K3']

E18.a.4 Prerequisites:18.2,10.j.1,10.j.12,18.a.3,18.a.2.Difficulty:**.Note:*A version of Ascoli–Arzelà's theorem.* [1K4]

Let $I \subseteq \mathbb{R}$ be a closed and bounded interval. Let $C(I)$ be the set of continuous functions $f : I \to \mathbb{R}$. We equip $C(I)$ with distance $d_\infty(f, g) = \|f - g\|_\infty$. We know that metric space $(C(I), d_\infty)$ is complete.

Let $\mathcal{F} \subseteq C(I)$: the following are equivalent.

1. $\mathcal{F}$ is compact
2. $\mathcal{F}$ is closed, it is equicontinuous and bounded (i.e. $\sup_{f\in\mathcal{F}} \|f\|_\infty < \infty$).

§19 Power series [1K6]

All definitions and theorems needed to solve the following exercises may be found in Chap. 6 of [2], Sec. 11.6 in [4], or Chap. 8 of [22].

Exercises

E19.1 A power series $\sum_{k=0}^{\infty} a_k x^k$ has a positive convergence radius, if and only if, $\exists \ell > 0$ for which $|a_k| \le \ell^k$ for every $k \ge 1$. [1K7]

Hidden solution: [UNACCESSIBLE UUID '1K8']

E19.2 Let c_k be complex numbers, and $a_k = |c_k|$. Note that power series $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} c_k z^k$ have the same radius of convergence R. [1K9]

Setting, for $t > 0$ real, $\tilde{f}(t) = \sum_{k=0}^{\infty} a_k t^k$, note that this formula defines a monotonic function $\tilde{f} : [0, \infty) \to [0, \infty]$; show that the radius of convergence R coincides with the upper bound of $t \ge 0$ such that $\tilde{f}(t) < \infty$.

Hidden solution: [UNACCESSIBLE UUID '1KB'] *Hidden solution:* [UNACCESSIBLE UUID '1KC']

E19.3 Prerequisites:7.e.4.Given $f(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_k \ge 0$, such that the radius of convergence is $r > 0$, show that $\lim_{t \to r-} f(t) = f(r)$. *Hidden solution:* [UNACCESSIBLE UUID '1KF'] [1KD]

E19.4 Find two examples of series $f(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_k > 0$ and with radius of convergence r positive and finite, such that [1KG]

- $f(r) < \infty$
- $f(r) = \infty$

Hidden solution: [UNACCESSIBLE UUID '1KH']

E19.5 Find an example of a series $f(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_k \in \mathbb{R}$ and with radius of convergence r positive and finite, such that the limit $\lim_{t \to r-} f(t)$ exists and is finite, but the series does not converge in $t = r$. [1KJ]

Hidden solution: [UNACCESSIBLE UUID '1KK']

Note that (by Abel's lemma) if the series converges in $t = r$ then the limit $\lim_{t \to r-} f(t)$ exists and $\lim_{t \to r-} f(t) = f(r)$.

E19.6 Let $b \in \mathbb{R}, n \in \mathbb{N}$. Assuming that $f(t) = \sum_{k=0}^{\infty} a_k t^k$ with radius of convergence r positive and $t \in (-r, r)$, determine the coefficients a_k so as to satisfy the following differential equations., [1KM]

1. $f'(t) = f(t)$ and $f(0) = b$,
2. $f'(t) = t^2 f(t)$ and $f(0) = b$,
3. $f''(t) = t^2 f(t)$ and $f(0) = b, f'(0) = 0$,
4. $t f''(t) + f'(t) + t f(t) = 0$ and $f(0) = b, f'(0) = 0$,
5. $t^2 f''(t) + t f'(t) + (t^2 - m^2) f(t) = 0$ $m \ge 2$ integer, $f(0) = f'(0) = \dots f^{(m-1)} = 0$, and $f^{(m)} = b$.

(The last two are called *Bessel equations*). *Hidden solution:* [UNACCESSIBLE UUID '1KP'] [UNACCESSIBLE UUID '27G']

See also the exercises 20.3, 24.a.2, 24.a.4 and 24.a.1.

§19.a Sum and product, composition and inverse [2D6]

Exercises

E19.a.1 `Prerequisites:19.2.`Consider power series [1KQ]

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad g(x) = \sum_{m=0}^{\infty} b_m x^m \ ,$$

with non-zero radius of convergence, respectively r_f and r_g.

Show that the product function $h(x) = f(x)g(x)$ can be expressed in power series

$$h(x) = \sum_{k=0}^{\infty} c_k x^k$$

where

$$c_k = \sum_{j=0}^{k} a_j b_{k-j} \ ;$$

with radius of convergence $r_h \geq \min\{r_f, r_g\}$. (Note the similarity with Cauchy's product, discussed in section §7.c.c)

Can it happen that $r_h > \min\{r_f, r_g\}$?

Hidden solution: *[UNACCESSIBLE UUID '1KR']*

E19.a.2 `Prerequisites:19.1.Difficulty:*.`Let $g(z) = \sum_{m=0}^{\infty} b_m z^m$ with $b_0 = g(0) \neq 0$. Express formally the reciprocal function $f(x) = 1/g(x)$ as a power series and calculate the coefficients starting from the coefficients b_m. If the radius of convergence of g is non-zero show that the radius of convergence of f is non-zero and that $f(x) = 1/g(x)$ where the two series $f(x), g(x)$ converge. *Hidden solution:* *[UNACCESSIBLE UUID '1KT']* [1KS]

E19.a.3 `Prerequisites:19.2,17.a.3.Difficulty:*.` [1KV]

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad g(x) = \sum_{m=0}^{\infty} b_m x^m \ ,$$

with non-zero radius of convergence, respectively r_f and r_g. Suppose $g(0) = 0 = b_0$. Let $I_f, I_g \subset \mathbb{C}$ be disks centered in zero with radii less than r_f and r_g, respectively: the previous series therefore define functions $f : I_f \to \mathbb{C}$ and $g : I_g \to \mathbb{C}$. Up to shrinking I_g, we assume that $g(I_g) \subset I_f$.

Show that the composite function $h = f \circ g : I_g \to \mathbb{C}$ can be expressed as a power series $h(x) = \sum_{k=0}^{\infty} c_k x^k$ (with radius of convergence at least r_g). Show how coefficients c_k can be computed from coefficients a_k, b_k. *Hidden solution:* *[UNACCESSIBLE UUID '1KW'][UNACCESSIBLE UUID '1KX'] [UNACCESSIBLE UUID '1KY']*

E19.a.4 `Difficulty:*.`Let $g(z) = \sum_{m=0}^{\infty} b_m z^m$ with non-zero radius of convergence r_g. Let $I_g \subset \mathbb{C}$ be a zero-centered disk of radius less than r_g; so we defined a function $g : I_g \to \mathbb{C}$. We assume $g(0) = 0$ and $g'(0) \neq 0$. Assuming that the inverse $f(y) = g^{-1}(y)$ can be expressed in Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, compute the coefficients of the series of f starting from those of g. [1KZ]

Hidden solution: *[UNACCESSIBLE UUID '1M0']*

E19.a.5 Prerequisites:19.a.4.Difficulty:**. [1M1]

Defining $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where the coefficients a_n were derived in the previous exercise 19.a.4, Try to show that the radius of convergence f is positive. †107

§19.b Exp,sin,cos [2D7]

Exercises

E19.b.1 Prerequisites:19.2,19.a.1, 6.7, 6.8.It is customary to define [1M3]

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

for $z \in \mathbb{C}$. We want to reflect on this definition.

- First, for each $z \in \mathbb{C}$, we can actually define

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

  (Note that the radius of convergence is infinite — as it easily occurs using the root criterion 7.c.1).
- We note that $f(0) = 1$; we define $e = f(1)$ which is *Euler's number*†108
- Show that $f(z + w) = f(z)f(w)$ for $z, w \in \mathbb{C}$.
- It is easy to verify that $f(x)$ is monotonic increasing for $x \in (0, \infty)$; by the previous relation, $f(x)$ is monotonic increasing for $x \in \mathbb{R}$.
- Then show that, for $n, m > 0$ integer, $f(n/m) = e^{n/m}$ (for the definition of $e^{n/m}$ see 6.7).
- Deduce that, for every $x \in \mathbb{R}$, $f(x) = e^x$ (for the definition of e^x see 6.8)

Hidden solution: *[UNACCESSIBLE UUID '1M4']*

E19.b.2 Prerequisites:7.e.4. [1M5]

Given $z \in \mathbb{C}$, show that

$$\lim_{N\to\infty} \left(1 + \frac{z}{N}\right)^N = e^z \tag{19.b.3}$$

and that the limit is uniform on compacts sets. *Hidden solution:* *[UNACCESSIBLE UUID '1M6']*

E19.b.4 If $z = x + iy$ with $x, y \in \mathbb{R}$, then we can express the complex exponential as a product $e^z = e^x e^{iy}$. Use power series developments to show *Euler's identity* $e^{iy} = \cos y + i \sin y$. [1M7]

Hidden solution: *[UNACCESSIBLE UUID '1M8']*

E19.b.5 Conversely, note then that $\cos y = \frac{e^{iy}+e^{-iy}}{2}$, $\sin y = \frac{e^{iy}-e^{-iy}}{i2}$. [1M9]

E19.b.6 Use the above formula to verify the identities [1MB]

$$\sin(x + y) = \cos x \sin y + \cos y \sin x$$

$$\cos(x + y) = \cos x \cos y - \sin y \sin x$$

Hidden solution: [UNACCESSIBLE UUID '1MC']

E19.b.7 We define the functions *hyperbolic cosine* †109 [1MD]

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

and *hyperbolic sine*

$$\sinh y = \frac{e^y - e^{-y}}{2}.$$

- Verify that
$$(\cosh x)^2 - (\sinh x)^2 = 1$$
(which justifies the name of "hyperbolic").
- Prove the validity of these power series expansion
$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \ldots$$
$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \ldots$$
- Check that
$$\cosh' = \sinh \ , \ \sinh' = \cosh$$
- Check the formulas
$$\sinh(x + y) = \cosh x \sinh y + \cosh y \sinh x$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh y \sinh x \ .$$

§19.c Matrix exponential [2D8]

Definition 19.c.1. *We define the exponential of matrices as* [1MF]

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where we agree that $A^0 = \mathbb{I}$, the identity matrix.

Exercises

E19.c.2 Prerequisites:Section . §12.d,12.c.3, 12.e.4, 12.d.4, 19.b.2. [1MG]

We equip the space of the matrices $\mathbb{C}^{n\times n}$ with one of the norms seen in Section §12.e.

†107 The proof can be found in Proposition 9.1 on pg 26 in Henri Cartan's book [7].
†108 Known as *numero di Nepero* in Italy.
†109 See Wikipedia page "Derivazione delle funzioni iperboliche" [37] which explains in what sense y is an "angle".

- Show that the series $\sum_{k=0}^{\infty} A^k/k!$ converges.
- Show that

$$\exp(A) = \lim_{N\to\infty} \left(\mathbb{I} + A/N\right)^N \tag{19.c.3}$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$; and that convergence is uniform in every compact neighborhood of A. *(Hint: make good use of the similar result 19.b.2.)*

Hidden solution: *[UNACCESSIBLE UUID '1MH']*

E19.c.4 If A is invertible then [1MJ]

$$A\exp(B)A^{-1} = \exp(ABA^{-1}) \quad .$$

E19.c.5 The derivative of [1MK]

$$t \in \mathbb{R} \mapsto \exp(tA)$$

is $A\exp(tA)$. *Hidden solution:* *[UNACCESSIBLE UUID '1MM']*

E19.c.6 If A, B commute, then [1MN]

$$A\exp(B) = \exp(B)A \quad , \quad \exp(A+B) = \exp(A)\exp(B) \, .$$

In particular $\exp(A)$ is always invertible and its inverse is $\exp(-A)$.

Hidden solution: *[UNACCESSIBLE UUID '1MP']*

E19.c.7 Let [1MQ]

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad :$$

compute

$$\exp(A)\exp(B) \quad , \quad \exp(B)\exp(A) \quad , \quad \exp(A+B) \quad ;$$

You will get that they are all different from each other. *Hidden solution:* *[UNACCESSIBLE UUID '1MR']*

E19.c.8 If A, B then the directional derivative of exp at the point A in the direction B is $B\exp(A)$, i.e. [1MS]

$$\frac{d}{dt}\exp(A+tB)|_{t=0} = B\exp(A) \quad .$$

E19.c.9 `Difficulty:*`.Show that [1MT]

$$\det(\exp(A)) = \exp(\operatorname{tr}(A)) \quad .$$

Hint: use Jacobi's formula 24.14 to calculate the derivative of $\det(\exp(tA))$*. Use the previous result 19.c.5 — see also 23.f.4. Another proof can be obtained by switching to Jordan's form (using 19.c.4).*

Hidden solution: *[UNACCESSIBLE UUID '1MV']*

E19.c.10 `Difficulty:*`.In the general case (when we do not know if A, B commute) we proceed as follows. Let's define $[A, B] = AB - BA$. [1MW]

- Setting $B_0 = B$ and $B_{n+1} = [A, B_n]$ you have

$$\begin{aligned} B_n &= A^n B - nA^{n-1}BA + \frac{n(n-1)}{2}A^{n-2}BA^2 + \cdots + (-1)^n BA^n = \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{n-k}BA^k \ ; \end{aligned}$$

- let's define now $Z = Z(A, B)$

$$Z \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{B_n}{n!} \ , \tag{19.c.11}$$

(note that Z is linear in B): prove that the above series converges, and that

$$\exp(A)B\exp(-A) = Z \ ; \tag{19.c.12}$$

- from this finally it is shown that

$$\exp(A)\exp(B)\exp(-A) = \exp(Z) \ .$$

(These formulas can be seen as consequences of the Baker–Campbell–Hausdorff formula [48]). Hidden solution: [UNACCESSIBLE UUID '1MX']

E19.c.13 Prerequisites:19.c.2.In general (even when A, B do not commute) [1MY]

$$\exp(A+B) = \lim_{N\to\infty} \Big(\exp(A/N)\exp(B/N)\Big)^N$$

Hidden solution: [UNACCESSIBLE UUID '1MZ']

E19.c.14 Prerequisites:23.f.3.Difficulty:**.In the general case (even when we do not know if A, B commute), we can express $\exp(A + sB)$ using a power series. Define [1N0]

$$C(t) = \exp(-tA)B\exp(tA)$$

and (recursively) set $Q_0 = \mathbb{I}$ (the identity matrix) and then

$$Q_{n+1}(t) = \int_0^t C(\tau)Q_n(\tau)\,d\tau$$

then

$$\exp(-A)\exp(A+sB) = \sum_{n=0}^{\infty} s^n Q_n(1) \ ; \tag{19.c.15}$$

this series converges for every s.

In particular, the directional derivative of exp at the point A in the direction B is

$$\frac{d}{ds}\exp(A+sB)|_{s=0} = \exp(A)Q_1(1) = \int_0^1 \exp((1-\tau)A)B\exp(\tau A)\,d\tau \ .$$

(Hint: Use the exercise 23.f.3 with $Y(t,s) = \exp(-tA)\exp(t(A+sB))$ and then set $t = 1$.)

Hidden solution: [UNACCESSIBLE UUID '1N1']

E19.c.16 Prerequisites:19.c.10.Difficulty:*. [1N2]

Prove the relations

$$\frac{d}{dt}\exp(A+tB)|_{t=0} = \int_0^1 \exp(sA)B\exp((1-s)A)\,\mathrm{d}s \ .$$

using the relations (19.c.11) and (19.c.12) from exercise 19.c.10.

§20 Analytic functions [1N4]

All definitions and theorems needed to solve the following exercises, may be found in Chap. 6 of [2] or Chap. 8 of [22].

Exercises

E20.1 Prerequisites:17.a.4. [1N5]

Verify that the function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} e^{-1/x} & se\ x > 0 \\ 0 & se\ x \le 0 \end{cases}$$

(also seen in 17.a.4) is not analytic.

Hidden solution: [UNACCESSIBLE UUID '1N6']

E20.2 Note:*Exercise 2, written exam March 2010.* [1N7]

Let $I \subseteq \mathbb{R}$ be a not-empty open interval. Let $f : I \to \mathbb{R}$ be of class C^∞, and such that $\forall x \in I, \forall k \ge 0$ we have $f^{(k)}(x) \ge 0$. Prove that f is analytic.

Hidden solution: [UNACCESSIBLE UUID '1N9'] See also the exercise 17.a.14.

E20.3 Prerequisites:19.a.1. [1NC]

Show that $f(x) = \frac{1}{1+x^2}$ is analytic on all $\mathbb{R}$, but the radius of convergence of the Taylor seried centered in x_0 is $\sqrt{1+x_0^2}$.

Hidden solution: [UNACCESSIBLE UUID '1ND'][UNACCESSIBLE UUID '1NF']

Study similarly $f(x) = \sqrt{x^2+1}$ or $f(x) = e^{1/(x^2+1)}$.

E20.4 Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ class function; fix $x_0 \in \mathbb{R}$ and define [1NG]

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

using the Taylor series; suppose g has radius of convergence $R > 0$: So $g : J \to \mathbb{R}$ is a well-defined function, where $J = (x_0 - R, x_0 + R)$. Can it happen that $f(x) \ne g(x)$ for a point $x \in J$?

And if f is analytic? [†110]

Hidden solution: [UNACCESSIBLE UUID '1NH']

E20.5 Let $I \subseteq \mathbb{R}$ be a nonempty open interval. Let $f : I \to \mathbb{R}$ be a C^∞ class function. Let [1NJ]

$$b_n = \sup_{x \in I} |f^{(n)}(x)| = \|f^{(n)}\|_\infty \quad ;$$

if

$$\limsup_{n\to\infty} \frac{1}{n} \sqrt[n]{b_n} < \infty$$

[†110] By "analytic" we mean: fixed x_0 there is a series $h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ with non-zero radius of convergence such that $f = h$ in an open neighborhood of x_0 (neighborhood contained in the convergence disk) .

then f is analytic.

Show with a simple example that the request is not necessary.

Hidden solution: *[UNACCESSIBLE UUID '1NK'][UNACCESSIBLE UUID '1NM']*

E20.6 Note:*Exercise 1, written exam, June 30th, 2017.* [1NN]

Let f be a continuous function on the interval $[0,1]$. Prove that the function

$$F(t) = \int_0^1 f(x) e^{tx} \, dx$$

is analytic on $\mathbb{R}$.

Hidden solution: *[UNACCESSIBLE UUID '1NP']*

E20.7 Let $I = (0,1)$, find an example of an analytic function $f : I \to \mathbb{R}$ not identically zero, but such that $A = \{x \in I : f(x) = 0\}$ has an accumulation point in $\mathbb{R}$. Compare this example with Prop. 6.8.4 in the notes [2]; and with the example 17.a.10. [1NQ]

Hidden solution: *[UNACCESSIBLE UUID '1NR']*

§21 Curve [1NT]

Let (X, d) be a metric space.

Definition 21.1. *Let $I \subseteq \mathbb{R}$ be an interval.* [1NV]

- *A continuous function $\gamma : I \to X$ is called* ***parametric curve****, or more simply in the following* curve.
- *If γ is injective, the curve is said to be* ***simple****.*
- *If γ is a homeomorphism onto its image, the curve is said to be* ***embedded****.*
- *If $X = \mathbb{R}^n$ and γ is of class C^1 and $\gamma'(t) \neq 0$ for every $t \in I$, then γ is called an* ***immersed curve*** *or* ***regular curve****.*

We will call ***support*** *or* ***trace*** *the image $\gamma(I)$ of a curve.*
The term arc *is also used as a synonym for* curve*;* [†111] *this term is mainly used when the curve is not (necessarily) closed.*

We postpone the study of closed curves to the next section.
Here are two notions of equivalence of curves. The first was taken from an earlier version of the the lecture notes [2].

Definition 21.2. *Let $I, J \subseteq \mathbb{R}$ be intervals. Let $\gamma : I \to X$ and $\delta : J \to X$ be two curves. We will write $\gamma \sim \delta$ if there exists an increasing homeomorphism* [†112] *$\varphi : I \to J$ such that $\gamma = \delta \circ \varphi$.* [1NW]

The second is Definition 7.5.4 from chapter 7 section 6 in the notes [2].

Definition 21.3. *Let $I, J \subseteq \mathbb{R}$ be intervals. Let $\gamma : I \to \mathbb{R}^n$ and $\delta : J \to \mathbb{R}^n$ be two regular curves. We will write $\gamma \approx \delta$ if there is a diffeomorphism* [†113] *$\varphi : I \to J$ monotonic increasing, such that $\gamma = \delta \circ \varphi$.* [1NX]

Exercises

E21.4 `Prerequisites:21.2,21.3.` [1J8]

Show that the relation $\gamma \sim \delta$ is an equivalence relation.

Show that the relation $\gamma \approx \delta$ is an equivalence relation.

E21.5 Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a function. Show that f is continuous if and only if, for each curve $\gamma : [0, 1] \to A$ we have that $f \circ \gamma$ is continuous. *Hidden solution:* *[UNACCESSIBLE UUID '1NZ']* [1NY]

E21.6 Suppose I is a closed and bounded interval; use the exercise 10.j.4 to show that a simple arc $\gamma : I \to X$ is a homeomorphism with its image, so the curve is embedded. [1P0]

Is the result still true if I is not closed? What if I is not bounded?

E21.7 `Prerequisites:21.6.Difficulty:*.` [1P1]

[†111] Note that in the book [22] an *arc* is an injective curve.
[†112] See 8.g.2.
[†113] A diffeomorphism is a bijective function $\varphi : I \to J$ of class C^1, the inverse of which is class C^1; in particular φ' is never zero, and (when domain and codomain are intervals) it always has the same sign.

Fix a curve $\gamma : I \to \mathbb{R}^n$. We define in the following $\hat{I} = \{t \in \mathbb{R} : -t \in I\}$ and $\hat{\gamma} : \hat{I} \to \mathbb{R}^n$ via $\hat{\gamma}(t) = \gamma(-t)$.

We want to show that, in certain hypotheses, two curves have the same support if and only if they are equivalent.

- Let $\gamma, \delta : [0,1] \to \mathbb{R}^n$ be simple curves, but not closed, and with the same support. Show that if $\gamma(0) = \delta(t)$ then $t = 0$ or $t = 1$. In case $\gamma(0) = \delta(0)$, show that $\gamma \sim \delta$. If instead $\gamma(0) = \delta(1)$ then $\hat{\gamma} \sim \delta$.
- Let $\gamma, \delta : [0,1] \to \mathbb{R}^n$ be simple immersed curves, but not closed, and with the same support, and let $\gamma(0) = \delta(0)$: show that $\gamma \approx \delta$. If instead $\gamma(0) = \delta(1)$ then $\hat{\gamma} \approx \delta$.

(For the case of closed curves see 21.a.10)

Hidden solution: *[UNACCESSIBLE UUID '1P2']*

E21.8 Show that $[0,1]$ and $[0,1]^2$ are not homeomorphic. *Hidden solution:* *[UNACCESSIBLE UUID '1P4']* [1P3]

E21.9 Prerequisites:10.j.4, 21.8.Show that you can't find a curve $c : [0,1] \to [0,1]^2$ continuous and bijective. Therefore a curve $c : [0,1] \to [0,1]^2$ that is continuous and surjective cannot be injective; such as the Peano curve, the Hilbert curve. [1P5]

Hidden solution: *[UNACCESSIBLE UUID '1P6']*

E21.10 Note:*Nice formula taken from [67].* [1P7]

Let $S = S(0,1) \subseteq \mathbb{R}^n$ be the unit sphere $S = \{x : |x| = 1\}$. Let $v, w \in S$ with $v \neq w$ and $v \neq -w$; let $T = \arccos(v \cdot w)$ so that $T \in (0, \pi)$; then the geodesic (that is, the arc-parameterized minimal length curve) $\gamma(t) : [0,T] \to S$ connecting v to w inside S is

$$\gamma(t) = \frac{\sin(T-t)}{\sin(T)} v + \frac{\sin(t)}{\sin(T)} w \quad ,$$

and its length is T.
(You may assume that, when $v \cdot w = 0$ that is $T = \pi/2$, then the geodesic is $\gamma(t) = v\cos(t) + w\sin(t)$). *Hidden solution:* *[UNACCESSIBLE UUID '1P8']*

§21.a Closed curves

We add other definitions to those already seen in 21.1.

Definition 21.a.1. *Let (X,d) be a metric space. Let $I = [a,b] \subseteq \mathbb{R}$ be a closed and bounded interval. Let $\gamma : I \to X$ be a parametric curve.* [1PB]

- *If $\gamma(a) = \gamma(b)$ we will say that the curve is **closed**;*
- *we also say that the curve is **simple and closed** if $\gamma(a) = \gamma(b)$ and γ is injective when restricted to $[a,b)$.* [†114]
- *If $X = \mathbb{R}^n$ and γ is class C^1 and is closed, it is further assumed that $\gamma'(a) = \gamma'(b)$.*

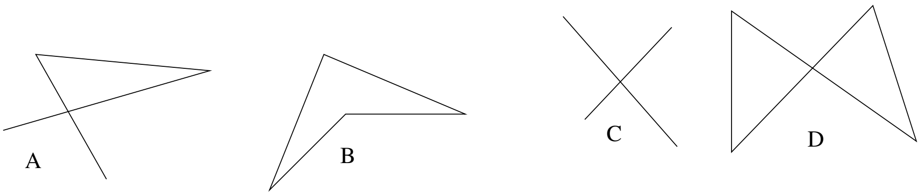


Figure 6: Sets for exercise 21.a.2

Exercises

E21.a.2 Consider the subsets of the plane described in the following figures 6: which can be the support of a simple curve? or a simple closed curve? or union of supports of two simple curves (possibly closed)? *(Prove your claims.)* [1PC]

E21.a.3 Let $\gamma : [0,1] \to X$ be a closed curve, show that it admits an extension $\tilde{\gamma} : \mathbb{R} \to X$ continuous and periodic with period 1. [1PF]

E21.a.4 Let $\gamma : [0,1] \to \mathbb{R}^n$ be a closed C^1 curve, show that it admits an extension $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^n$ periodic with period 1 and of class C^1. [1PG]

E21.a.5 We will use the definitions and results of the Section §10.o, in particular 10.o.7. [1PH]

Fix $\tilde{\gamma} : \mathbb{R} \to X$ continuous and periodic (of period 1); we can define the map $\hat{\gamma} : S^1 \to X$ through the relation

$$\hat{\gamma}\Big((\cos(t), \sin(t))\Big) = \tilde{\gamma}(t) \quad .$$

Show that this is a good definition, and that $\hat{\gamma}$ is continuous.

Use the exercise 10.j.4 to show that every closed simple arc, when viewed equivalently as a map $\hat{\gamma} : S^1 \to X$, is a homeomorphism with its image.

In the following we will use periodic maps to represent the closed curves.

Exercises

E21.a.6 Adapt the notion of equivalence 21.2 to the case of simple and closed arcs, but considering them as maps $\gamma : \mathbb{R} \to X$ continuous and periodic (of period 1); what hypotheses do we require from the maps $\varphi : \mathbb{R} \to \mathbb{R}$? [1PK]

Hidden solution: [UNACCESSIBLE UUID '1PM']

E21.a.7 `Prerequisites:21.2,21.a.3.`Let γ, δ be closed curves, but seen as maps defined on $\mathbb{R}$, continuous and periodic of period 1. [1PN]

Let's discuss a new relation: we write $\gamma \sim_f \delta$ if there is an increasing homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t+1) = \varphi(t) + 1$ for every $t \in \mathbb{R}$, and for which $\gamma = \delta \circ \varphi$

Show that this is an equivalence relation.

[†114] That is, the injectivity is lost in the extremes.

Compare it with the relation $\sim$.

Hidden solution: [UNACCESSIBLE UUID '1PP']

E21.a.8 Prerequisites:21.3,21.a.4.Let γ, δ curves be closed and immersed, but seen as maps defined on $\mathbb{R}$ and C^1 and periodic. with periods 1. [1PQ]

Let's see a new relation: you have $\gamma \approx_f \delta$ if there is an increasing diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t+1) = \varphi(t) + 1$ for every $t \in \mathbb{R}$ and for which $\gamma = \delta \circ \varphi$

Show that this is an equivalence relation.

Compare it with the relation $\approx$.

E21.a.9 Prerequisites:21.3,21.a.4,21.a.8.Give a simple example of closed curves immersed for which you have $\gamma \approx_f \delta$ but not $\gamma \approx \delta$. [1PR]

Hidden solution: [UNACCESSIBLE UUID '1PS']

E21.a.10 Prerequisites:21.6.Difficulty:*. [1PT]

Let $\gamma, \delta : S^1 \to \mathbb{R}^n$ be simple and immersed closed curves with the same support; Define $\hat\gamma(t) = \gamma(-t)$: show that either $\gamma \approx_f \delta$ or $\hat\gamma \approx_f \delta$.

Hidden solution: [UNACCESSIBLE UUID '1PV']

Other exercises regarding curves are 11.21, 15.a.24, 17.d.13 and 24.4; see also Section §23.d.

§22 Surfaces [1PZ]

Exercises

E22.1 Prerequisites:17.d.1.Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ in C^1. Fix $\overline{x} \in A$ such that $f(\overline{x}) = 0$, and $\nabla f(\overline{x}) \neq 0$: by the implicit function theorem 17.d.1 the set $E = \{f = 0\}$ is a graph in a neighborhood of $\overline{x}$, and the plane tangent to this graph is the set of x for which [1Q0]

$$\langle x - \overline{x}, \nabla f(\overline{x})\rangle = 0 \ .$$

Compare this result to Lemma 7.7.1 in the notes [2]: "the gradient is orthogonal to the level sets" . Hidden solution: *[UNACCESSIBLE UUID '1Q1']*

E22.2 Given $m > 0$, show that the relation $xyz = m^3$ defines a surface in $\mathbb{R}^3$. Prove that the planes tangent to the surface at the points of the first octant $\{x > 0, y > 0, z > 0\}$ form with the coordinate planes of $\mathbb{R}^3$ a tetrahedron of constant volume. [1Q2]

Hidden solution: *[UNACCESSIBLE UUID '1Q3']*

E22.3 Let $a > 0$. Show that the equation $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ defines a regular surface inside the first octant $\{x > 0, y > 0, z > 0\}$. Prove that planes tangent to the surface cut the three coordinate axes at three points, the sum of whose distances from the origin is constant. [1Q4]

Hidden solution: *[UNACCESSIBLE UUID '1Q5']*

E22.4 Fix $a > 0, b > 0, c > 0$. Determine a plane tangent to the ellipsoid [1Q8]

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

at a point with $x, y, z > 0$, so that the tetrahedron bounded by this plane and the coordinated planes has minimum volume.

Hidden solution: *[UNACCESSIBLE UUID '1Q9']*

§23 Ordinary Differential equations [1QB]

To solve the following exercises, it is imporant to know some fundamental results, such as: the existence and uniqueness theorem [†115], Gronwall's Lemma; and in general some methods to analyze, solve and qualitative study Ordinary Differential Equations (abbreviated ODE). These may be found *e.g.* in [25, 20, 2].

Exercises

E23.1 For each point (x, y) of the plane with $x, y > 0$ passes a single ellipses $4x^2+y^2 = a$ (with $a > 0$). Describe the family of curves that at each point are orthogonal to the ellipse passing through that point. See figure 7. [1QC]

Hidden solution: *[UNACCESSIBLE UUID '1QF']*

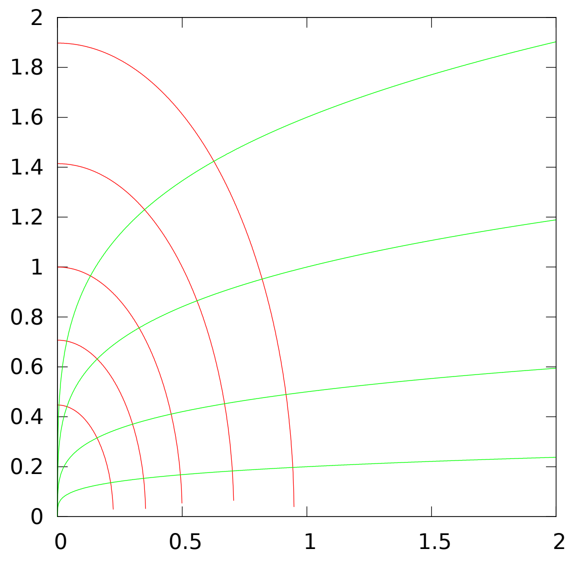


Figure 7: Ellipses (in red) and curves orthogonal to them.

E23.2 `Prerequisites:17.4.` [1QH]

Let $I \subseteq \mathbb{R}$ be an open interval.

Let $F : I \times \mathbb{R} \to (0, \infty)$ be a positive continuous function, and let $f : I \to \mathbb{R}$ be a differentiable function that solves the differential equation

$$(f'(x))^2 = F(x, f(x)) \quad .$$

Prove that x is, either always increasing, in which case $f'(x) = \sqrt{F(x, f(x))}$ for every x, or it is always decreasing, in which case $f'(x) = -\sqrt{F(x, f(x))}$; therefore f is of class C^1.

Hidden solution: *[UNACCESSIBLE UUID '1QJ']*

E23.3 `Prerequisites:23.2.` [1QK]

[†115] *A.k.a.* Picard–Lindelöf theorem, or Cauchy–Lipschitz theorem.

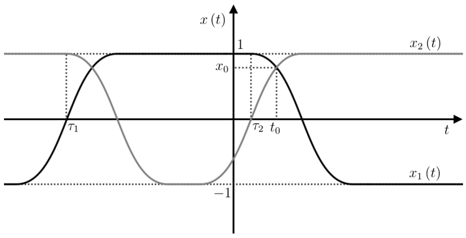


Figure 8: Figure for 23.3

Describe all the differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that solve

$$\forall x \,,\ (f'(x))^2 + (f(x))^2 = 1 \,.$$

Show that if $-1 < f(x) < 1$ for $x \in I$ open interval, then f is a sine arc, for $x \in I$.

Show that all solutions are C^1, and that they are piecewise C^∞.

Note that $f \equiv 1$ and $f \equiv -1$ are envelopes of the other solutions, as explained in the section §23.d.

Hidden solution: [UNACCESSIBLE UUID '1QM']

E23.4 Let $f : [0,1] \to \mathbb{R}$ be a function C^2 such that $f(0) = f(1) = 0$ and $f'(x) = f(x)f''(x)$ for every $x \in [0,1]$. [1QN]

Prove that the function f is identically zero.

Hidden solution: [UNACCESSIBLE UUID '1QP'][UNACCESSIBLE UUID '1QQ']

§23.a Autonomous problems

Exercises

E23.a.1 Prerequisites:16.3.Let su fix $x_0, t_0 \in \mathbb{R}$, and a bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$, with $f(x_0) = 0$ but $f(x) > 0$ for $x \neq x_0$. We want to study the autonomous problem [1QR]

$$\begin{cases} x'(t) = f(x(t)) \ , \\ x(t_0) = x_0 \ . \end{cases}$$

Note that $x \equiv x_0$ is a possible solution. Show that if, for $\varepsilon > 0$ small, [†116]

$$\int_{x_0}^{x_0+\varepsilon} \frac{1}{f(y)} \,\mathrm{d}y = \infty \tag{23.a.2}$$

$$\int_{x_0-\varepsilon}^{x_0} \frac{1}{f(y)} \,\mathrm{d}y = \infty \tag{23.a.3}$$

then $x \equiv x_0$ is the only solution; while otherwise there are many class C^1 solutions: describe them all.

Hidden solution: [UNACCESSIBLE UUID '1QS']

Conditions (23.a.2) and (23.a.3) are a special case of *Osgood uniqueness condition*, see Problem 2.25 in [25].

[†116] If the condition holds for a $\varepsilon > 0$ then it holds for every $\varepsilon > 0$, since $f > 0$ far from x_0.

E23.a.4 Set $\alpha > 1$ and consider [1QV]

$$\begin{cases} x'(t) = |x(t)|^\alpha \\ x(t_0) = x_0 \end{cases} ,$$

with $x_0, t_0 \in \mathbb{R}$ fixed. Show that there is existence and uniqueness of the solution; calculate the maximal definition interval; Use the variable separation method to explicitly calculate solutions. (Since the equation is autonomous, one could assume that $t_0 = 0$, but the example is perhaps clearer with a generic t_0).

Hidden solution: *[UNACCESSIBLE UUID '1QW']*

E23.a.5 What happens in the previous exercise in the case $\alpha \in (0, 1)$? [1QX]

Hidden solution: *[UNACCESSIBLE UUID '1QY']*

E23.a.6 Prerequisites:23.a.4.Let us fix $\alpha > 1$, and consider again [1QZ]

$$\begin{cases} x'(t) = |x(t)|^\alpha \\ x(0) = 1 \end{cases} ,$$

We have seen in 23.a.4 that this ODE admits a maximal solution $x : I_\alpha \to \mathbb{R}$. Fixed $t \in \mathbb{R}$, show that $t \in I_\alpha$ for $\alpha > 1$ close to 1, and that $\lim_{\alpha \to 1+} x(t) = e^t$.

Note that e^t is the only solution of $x'(t) = |x(t)|$ with $x(0) = 1$.

Hidden solution: *[UNACCESSIBLE UUID '1R0']*

§23.b Resolution

Exercises

E23.b.1 Let $\Theta : \mathbb{R} \to \mathbb{R}$ be a continuous function, Describe all solutions $f : \mathbb{R} \to \mathbb{R}$ that solve [1R1]

$$\forall x \neq 0 \, , \; f'(x) = \Theta\left(\frac{f(x)}{x}\right)$$

(Hint: change variables $f(x) = xh(x)$ and find and solve a differential equation for $h(x)$.)

Hidden solution: *[UNACCESSIBLE UUID '1R2']*

E23.b.2 Find solutions to the problem [1R4]

$$\frac{dy}{dx} = \frac{y}{x+y}$$

with substitution $z = y/x$, and also comparing it with the problem

$$\frac{dx}{dy} = \frac{x+y}{y} \, .$$

§23.c Qualitative discussions

For the following exercises the following simple comparison lemma may be useful.

Lemma 23.c.1. *Let $U \subseteq \mathbb{R}^2$ be open, let $f, g : U \to \mathbb{R}$ be continuous with $f \geq g$; let $I \subseteq \mathbb{R}$ be an open interval with $t_0 \in I$, and let $x, w : I \to \mathbb{R}$ solutions of* [1R7]

$$x'(t) = f(t, x(t)) \quad , \quad w(t) = g(t, w(t))$$

with $x(t_0) \geq w(t_0)$: then $x(t) \geq w(t)$ for $t \geq t_0$. Note indeed that $x'(t) \geq w'(t)$ and therefore $x(t) - w(t)$ is an increasing function.

(There are much more refined versions of this lemma, see for example in section 8.6 in the course notes [2]).

Exercises

E23.c.2 Discuss solutions of [1R8]

$$\begin{cases} y'(x) = (y(x) - x)^3 \\ y(0) = a \ . \end{cases}$$

Qualitatively study the existence (local or global) of solutions, and the properties of monotonicity and convexity/concavity.

Hidden solution: [UNACCESSIBLE UUID '1R9'][UNACCESSIBLE UUID '1RB']

E23.c.3 Considering the Cauchy problem [1RD]

$$\begin{cases} y'(x) = \frac{1}{y(x)^2 + x^2} \\ y(0) = 1 \end{cases}$$

show that there is only one global solution $y : \mathbb{R} \to \mathbb{R}$, that y is bounded, and the limits $\lim_{x\to\infty} y(x)$, $\lim_{x\to-\infty} y(x)$ exist and are finite.

Hidden solution: [UNACCESSIBLE UUID '1RG'][UNACCESSIBLE UUID '1RH']

E23.c.4 Discuss the differential equation [1RK]

$$\begin{cases} y'(x) = \frac{1}{y(x) - x^2} \\ y(0) = a \end{cases}$$

for $a \neq 0$, studying in a qualitative way the existence (local or global) of solutions, and the properties of monotonicity and convexity/concavity. †117

Show that the solution exists for all positive times.

Show that for $a > 0$ the solution does not extend to all negative times.

Difficulty:*. Show that there is a critical $\tilde{a} < 0$ such that, for $\tilde{a} < a < 0$ the solution does not extend to all negative times, while for $a \leq \tilde{a}$ the solution exists for all negative times; also for $a = \tilde{a}$ you have $\lim_{x\to-\infty} y(x) - x^2 = 0$.

Hidden solution: [UNACCESSIBLE UUID '1RP']

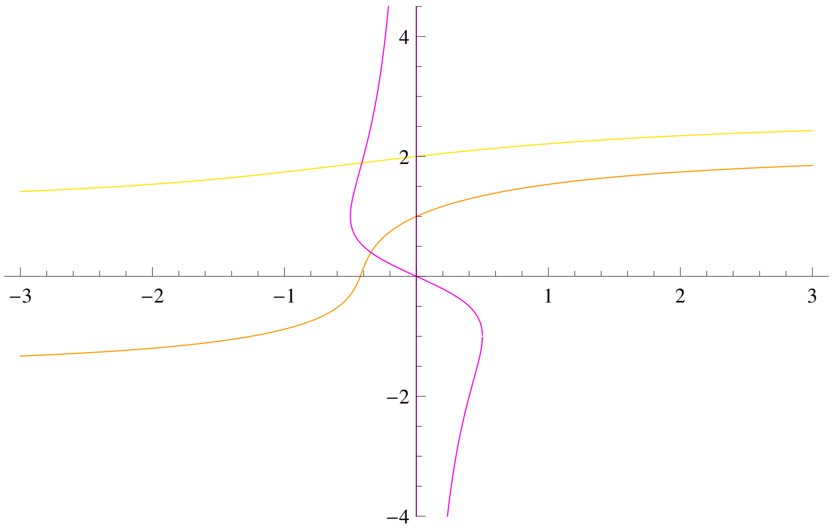


Figure 9: Exercise 23.c.3. In purple the line of inflections. In yellow the solutions with initial data $y(0) = 1$ and $y(0) = 2$.

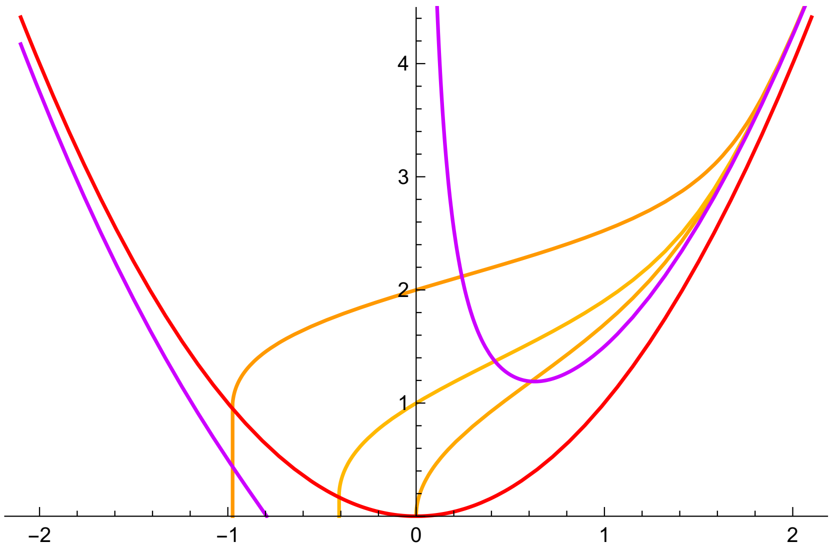


Figure 10: Exercise 23.c.4. Solutions for $a > 0$
In purple the line of inflections. In red the parabola where the derivative of the solution is infinite. In yellow the solutions with initial data $y(0) = 2$, $y(0) = 1$, $y(0) = 1/1000$.

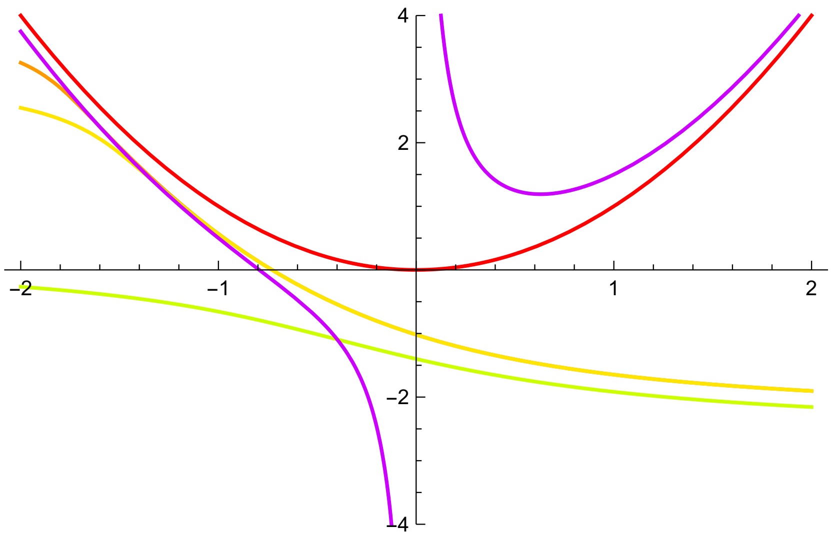


Figure 11: Exercise 23.c.4. Solutions for $a < 0$
In purple the line of inflections. In red the parabola where the derivative of the solution is infinite. Solutions are drawn with initial data $a = -1.4$ ("green"), $a = -1.0188$ ("orange") and $a = -1.019$ ("yellow"). Note that the latter two differ only by 0.0002 in their initial data (indeed they are indistinguishable in the graph for $x > -1$), but then for $x < -1$ they move apart quickly, and for $x = -2$ they are respectively 3.25696 and 2.54856, with a difference of about 0.7 !

E23.c.5 Note:*Exercise 4, written exam 9 July 2011.*Show that the Cauchy problem [1RQ]

$$\begin{cases} y'(x) = y(x)\big(y(x) - x^2\big) \\ y(2) = 1 \end{cases}$$

admits a single solution $y = y(x)$, defined on all of $\mathbb{R}$ and such that

$$\lim_{x\to-\infty} y(x) = +\infty \quad , \quad \lim_{x\to\infty} y(x) = 0 \quad .$$

†117 The differential equation is taken from exercise 13 in [1].

§23.d Envelope

Given a family of planar curves, we want to define the *envelope curve*. Let's see two possible definitions.

Definition 23.d.1 (Curve Envelope). [23Y]

- *Suppose the curves in the plane are described by the equation in implicit form $F(x, y, a) = 0$; that is, fixed the parameter a, the curve is the locus*
$$\{(x, y) : F(x, y, a) = 0\} \quad ;$$
Then the envelope is obtained by expliciting the variable a from the equation $\frac{\partial}{\partial a} F(x, y, a) = 0$ and substituting it into the $F(x, y, a) = 0$.
- *For simplicity, consider curves that are functions of the abscissa. Let $y = f(x, a) = f_a(x)$ be a family of functions, with $x \in I, a \in J$ (open intervals), then $y = g(x)$* ***is the envelope of*** *f_a if the graph of g is covered by the union of the graphs of f_a and the curve g is tangent to every f_a where it touches it. More precisely, for every $x \in I$ there is $a \in J$ for which $g(x) = f(x, a)$, and also, for every choice of a that satisfies $g(x) = f(x, a)$, we have $g'(x) = f'(x, a)$.*

Remark 23.d.2. *The envelope curve has an important property in the field of differential equations. Suppose $y = f_a(x)$ are solutions of the differential equation $\Phi(y', y, x) = 0$: then also g is solution (immediate verification).* †118 [240]

We want to see that the two previous definitions are equivalent in this sense.

Exercises

E23.d.3 Let's start with the first definition. Suppose we can apply the Implicit Function Theorem to the locus [1RV]

$$E_a = \{(x, a) : F(x, y, a) = 0\} \quad ;$$

Precisely, suppose that at a point $(\overline{x}, \overline{y}, \overline{a})$ we have that $\frac{\partial F}{\partial y} \neq 0$. To this we also add the hypothesis $\frac{\partial^2 F}{\partial aa} \neq 0$. Fixed a, you can express E_a locally as a graph $y = f(x, a) = f_a(x)$. We also use the hypothesis $\frac{\partial^2 F}{\partial aa} \neq 0$ to express locally $\frac{\partial F}{\partial a} = 0$ as a graph $a = \Phi(x, y)$. Defining $G(x, y) \stackrel{\text{def}}{=} F(x, y, \Phi(x, y))$, show that $G = 0$ can be represented as $y = g(x)$. Finally, show that g is the envelope of the curves f_a.

Hidden solution: *[UNACCESSIBLE UUID '1RW'][UNACCESSIBLE UUID '1RX']*

E23.d.4 In the above hypotheses, assuming that $\frac{\partial F}{\partial y} > 0$ and $\frac{\partial^2 F}{\partial aa} > 0$, show that the envelope graph g is locally the "edge" of the union of the graphs f_a (in the sense that $g(x) \geq f_a(x)$ with equality for only one a). [1RY]

Hidden solution: *[UNACCESSIBLE UUID '1RZ'][UNACCESSIBLE UUID '1S0']*

E23.d.5 Note:*From the text [19], pg 84.*.Consider the curves [1S1]

$$y = f(x, a) = ax + \frac{a^2}{2}$$

†118 With equations in normal form, however, this notion is not interesting because there is local uniqueness and then there can be no special solutions; that is, if $g = f_a$ $g' = f'_a$ at a point x then they coincide in a neighborhood.

- Find a differential equation solved by all curves. *(Sugg. Eliminate a from the system $y = f, y' = \frac{\partial}{\partial x} f$. The result can be left in non-normal form.)*
- Calculate the envelope; check that it satisfies the differential equation obtained above.

See also the figure 12. *Hidden solution:* [UNACCESSIBLE UUID '1S2']

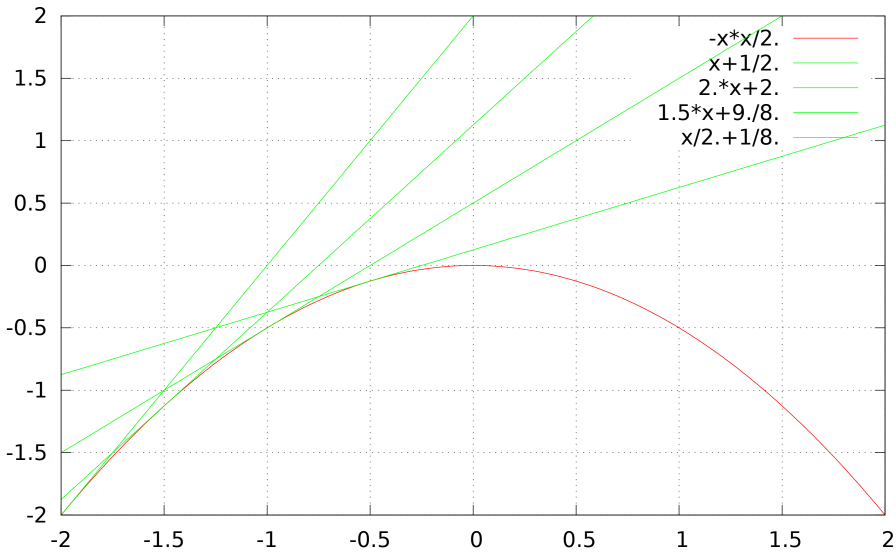


Figure 12: Solution of 23.d.5: envelope.

E23.d.6 Consider ellipses $ax^2 + y^2/a = 2$ (with $a > 0$). [1S4]

- Find the region of the plane covered by these ellipses.
- Show that the edge of this region is the envelope of ellipses, and describe it.

Hidden solution: [UNACCESSIBLE UUID '1S5'][UNACCESSIBLE UUID '1S6']

E23.d.7 Let's consider the lines $ax + y/a = 1$ (with $a > 0$). [1S7]

- Find the region of the first quadrant covered by these lines.
- Show that the edge of this region is the envelope of the lines and describe it.

Hidden solution: [UNACCESSIBLE UUID '1S8']

E23.d.8 Let's consider the straight lines [1S9]

$$\frac{x}{a} + \frac{y}{1-a} = 1$$

with $x, y, a \in (0, 1)$. Describe the envelope curve.

Hidden solution: [UNACCESSIBLE UUID '1SB']

§23.e Linear equations (with constant coefficients)

Definition 23.e.1. *We formally indicate with D the operation* "computing of the derivative". *Given a polynomial $p(x)$* [23Z]

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

(which has constants coefficients $a_i \in \mathbb{C}$) we formally construct the linear operator

$$p(D) = a_n D^n + a_{n-1}D^{n-1} + \dots a_1 D + a_0$$

which transforms a function $f : \mathbb{R} \to \mathbb{C}$ of class C^{n+k} into the function $p(D)f$, class at least C^k, defined pointwise by

$$[p(D)f](x) \stackrel{def}{=} a_n f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \dots a_1 f'(x) + a_0 f(x) \quad .$$

Exercises

E23.e.2 Given two polynomials $p(x), q(x)$ and the product polynomial $r(x) = p(x)q(x)$, show that $p(D)[q(D)f] = r(D)f$ [1SC]

E23.e.3 Define $f(x) = e^{\lambda x}$, note that [1SD]

$$[p(D)f](x) = p(\lambda)f(x) \quad .$$

We can therefore consider exponentials $e^{\lambda x}$ as eigenvectors of $p(D)$, with eigenvalue $p(\lambda)$.

E23.e.4 Let $f : \mathbb{R} \to \mathbb{C}$ be a C^n class function , let $\theta \in \mathbb{C}$ be a constant, and let $g(x) = e^{\theta x} f(x)$. Show that, if p is a polynomial and $q(x) = p(x + \theta)$, then [1SF]

$$p(D)g = e^{\theta x}[q(D)f] \quad .$$

Note that we can also write the relation above as a "conjugation"

$$e^{-\theta x}\big[p(D)[e^{\theta x} f]\big] = p(D + \theta)f \; .$$

Hidden solution: [UNACCESSIBLE UUID '1SG']

E23.e.5 Prerequisites:23.e.4.Given $\theta \in \mathbb{C}$ and $k \in \mathbb{N}$, define $p(x) = (x - \theta)^k$, show that $p(D)f = 0$ if and only if $f(x) = e^{\theta x} r(x)$ with r polynomial of degree at most $k - 1$. [1SH]

Hidden solution: [UNACCESSIBLE UUID '1SJ']

E23.e.6 Prerequisites:16.1, 23.e.4. [1SK]

Fix $\theta, \tau \in \mathbb{C}$ with $\theta \neq \tau$, $q(x)$ a polynomial, and $k \in \mathbb{N}$. Let's define $p(x) = (x-\theta)^k$. Show that

$$p(D)f(x) = e^{\tau x} q(x)$$

if and only if

$$f(x) = e^{\theta x} r(x) + e^{\tau x}\tilde{q}(x) \, ,$$

with r polynomial of degree at most $k - 1$ and $\tilde{q}$ polynomial of the same degree as q.

Hidden solution: [UNACCESSIBLE UUID '1SM']

E23.e.7 Given $a_0 \dots a_n \in \mathbb{C}$ constants, with $a_n \neq 0$, and defining $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$, describe all possible solutions f of [1SN]

$$p(D)f = 0\ .$$

Show that the solution space is a vector space (based on the field $\mathbb{C}$ of complex numbers) of dimension n.

(Hint. Factorize the polynomial and take advantage of previous exercises.).

E23.e.8 Prerequisites:23.e.7.With p as above, also analyze the problem [1SP]

$$p(D)f = e^{\alpha x}$$

(with $\alpha \in \mathbb{C}$ constant).

What happens when α approaches a root of the polynomial p?

[UNACCESSIBLE UUID '1SQ']Given parameters $y_0, \dots, y_{n-1} \in \mathbb{C}$, and also $\alpha \in \mathbb{C}$, the solution of the Cauchy problem [1SR]

$$\begin{cases} p(D)f = e^{\alpha x} \\ f(0) = y_0, \\ \dots \\ f^{n-1}(0) = y_{n-1} \end{cases}$$

exists for all times, and depends continuously on the parameters $\alpha, y_0, \dots, y_{n-1} \in \mathbb{C}$.

E23.e.9 Given $h = h(x)$, and $\theta \in \mathbb{R}$, solve the differential equations [1SS]

$$(D - \theta)f(x) = h(x)$$

$$(D - \theta)^2 f(x) = h(x)$$

$$(D^2 + \theta^2)f(x) = h(x)$$

$$(D^2 - \theta^2)f(x) = h(x)$$

and special cases

$$(D - 1)f(x) = x^k$$

$$(D - \theta)f(x) = e^{\alpha x}$$

(with $\alpha \in \mathbb{C}$, and $k \in \mathbb{N}$, constants).

Hidden solution: *[UNACCESSIBLE UUID '1SV']*

§23.f Matrix equations

To solve the following exercises you need to know the elementary properties of the exponential of matrices, see section §19.c.

Exercises

E23.f.1 Prerequisites:19.c.6,19.c.5, Section §19.c. [1SW]

Given $A, C \in \mathbb{C}^{n\times n}$ and $F : \mathbb{R} \to \mathbb{C}^{n\times n}$ continuous matrix valued functions, solve the ODE

$$X' = AX + F \quad, X(0) = C \quad ,$$

where $X : \mathbb{R} \to \mathbb{C}^{n\times n}$.

(Hint: use the method of variation of constants: replace $Y(t) = \exp(-tA)X(t)$ *)*

Hidden solution: *[UNACCESSIBLE UUID '1SX']*

E23.f.2 Prerequisites:19.c.6,19.c.5, Sec. §19.c.Difficulty:*. [1SY]

Given matrixes $A, B, C \in \mathbb{C}^{n\times n}$, solve the ODE

$$X' = AX + XB \quad, X(0) = C \quad ,$$

where $X : \mathbb{R} \to \mathbb{C}^{n\times n}$

Hidden solution: *[UNACCESSIBLE UUID '1SZ']*

E23.f.3 Prerequisites:12.c.3,12.e.5.Difficulty:*. [1T1]

Let $V = \mathbb{C}^{n\times n}$ a matrix space, we equip it with a submultiplicative norm $\|C\|_V$. Let $C \in V$ and let $A, B : \mathbb{R} \to V$ be continuous curves in space of matrices.

- We recursively define $Q_0 = C$, and

  $$Q_{n+1}(s) = \int_0^s A(\tau)Q_n(\tau)B(\tau)\,\mathrm{d}\tau \quad ;$$

  show that the series

  $$Y(t) = \sum_{n=0}^{\infty} Q_n(t)$$

  is well defined, showing that, for every $T > 0$, it converges totally in the space of continuous functions $C^0 = C^0([-T, T] \to V)$, endowed with the norm

  $$\|Q\|_{C^0} \stackrel{\text{def}}{=} \max_{|t|\le T} \|Q(t)\|_V \quad .$$

- Show that the function just defined is the solution of the differential equation

  $$\frac{d}{dt}Y(t) = A(t)Y(t)B(t) \quad , \quad Y(0) = C \quad .$$

- If A, B are constant, note that

  $$Y(t) = \sum_{n=0}^{\infty} t^n \frac{A^n C B^n}{n!} \quad .$$

Hidden solution: [UNACCESSIBLE UUID '1T2']

E23.f.4 Prerequisites:23.f.3,23.f.4.Note:*Abel's identity*. [1T3]

Let be given $C \in \mathbb{C}^{n\times n}$, $A : \mathbb{R} \to \mathbb{C}^{n\times n}$ continuous, and the solution $Y(t)$ of the ODE

$$\frac{d}{dt}Y(t) = A(t)Y(t) \quad , \quad Y(0) = C$$

(which has been studied in 23.f.3). Set $a(t) = \mathrm{tr}(A(t))$, show that

$$\det(Y(t)) = \det(C)e^{\int_0^t a(\tau)\,d\tau} \quad .$$

If C is invertible, it follows that $Y(t)$ is always invertible.

Hidden solution: [UNACCESSIBLE UUID '1T4']

E23.f.5 Prerequisites:19.c.6,19.c.5,23.f.3. [1T6]

Let be given $C \in \mathbb{C}^{n\times n}$, $F, A : \mathbb{R} \to \mathbb{C}^{n\times n}$ continuous, and the solution $Y(t)$ of the ODE

$$\frac{d}{dt}Y(t) = A(t)Y(t) \quad , \quad Y(0) = \mathrm{Id} \ .$$

Solve the equation

$$X' = AX + F \ \ , X(0) = C \ \ ,$$

where $X : \mathbb{R} \to \mathbb{C}^{n\times n}$, using $Y(t)$ as an auxiliary function.

Hidden solution: [UNACCESSIBLE UUID '1T7']

§24 Written exams and pseudo–exams [1T8]

Exercises

E24.1 Note:*reworked from the written exam held January 26th, 2016.* [1T9]

Let $(q_n)_{n\geq 1}$ be an enumeration of the rationals of $(0,1)$ and define

$$f(t) \stackrel{\text{def}}{=} \sum_{n:q_n<t} 2^{-n}$$

and

$$g(t) \stackrel{\text{def}}{=} \sum_{n:q_n\leq t} 2^{-n}$$

for $t \in (0,1)$.

- Show that f, g are strictly increasing.
- Calculate limits for $t \downarrow 0$ and $t \uparrow 1$.
- Show that f is left continuous, g is right continuous, and that

$$\lim_{\tau\to t+} f(\tau) = g(t) \quad , \quad \lim_{\tau\to t-} g(\tau) = f(t) \quad .$$

- Also show that f is discontinuous in t if and only if $t \in \mathbb{Q}\cap(0,1)$; and similarly for g.
- What changes if we replace 2^{-n} with the term a_n of an absolutely convergent series?

Hidden solution: *[UNACCESSIBLE UUID '1TB'][UNACCESSIBLE UUID '1TC']*

E24.2 Prerequisites:14.a.9.Note:*written exam, June 23th, 2012.* [1TD]

Let f be a C^1 class function on $\mathbb{R}$, with $f(0) \neq 0$. Prove that $x \in \mathbb{R}$ exists such that the two vectors

$$v = (x, f(x)) \quad , \quad w = (-f'(x), 1)$$

are linearly dependent. (Note that the vector w is orthogonal to the tangent of the graph of f.) Discuss the possibility that this condition is verified for every $x \in \mathbb{R}$.

Hidden solution: *[UNACCESSIBLE UUID '1TF']*

E24.3 Note:*adapted from the written exam, April 9th, 2011.* [1TG]

Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to+\infty} f(x)/x = +\infty \quad .$$

- Fixed $a < f(0)$, let M_a be the set of $m \in \mathbb{R}$ such that the line $y = mx + a$ intersects the graph $y = f(x)$ of the function f at least in one point: show that M_a admits minimum $\hat{m} = \hat{m}(a)$;
- show that $\hat{m}$ depends continuously on a, [†119]
- and that $\hat{m}(a)$ is monotonic strictly decreasing.

[†119] Tip: Rethink the exercise 14.a.9.

- If f is differentiable, show that the line $y = \hat{m}(a)x + a$ is tangent to the graph at all points where it encounters it.
- Suppose further that f is of class C^2 and that $f''(x) > 0 \forall x > 0$ [†120]. Show that there is only one point x where the line $y = \hat{m}(a)x + a$ meets the graph $y = f(x)$; name it $\hat{x} = \hat{x}(a)$;
- and show that the functions $a \mapsto \hat{x}(a)$ and $a \mapsto \hat{m}(a)$ are differentiable.

Hidden solution: [UNACCESSIBLE UUID '1TH']

E24.4 Topics:osculating circle. Note:*adapted from the written exam, April 9th 2011.* [1TJ]

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable in 0, with $f(0) = 0$ and $f''(0) \neq 0$. Prove that there is an unique point $P = (a, b)$ in the plane and an unique constant $r > 0$, such that

$$d\big(P, (x, f(x))\big) = r + o(x^2),$$

determining a, b, r as a function of $f'(0), f''(0)$. Here $d(P, Q)$ is the Euclidean distance between two points P, Q in the plane.

Hint. First, study the case in which also $f'(0) = 0$.

(The graph of the function f is a curve in the plane; by hypothesis this curve passes through the origin. In this exercise we have determined the circle, of radius r and center P, which best approximates the curve near the origin. This circle is called the "osculating circle", and its radius is called the "radius of curvature", and the inverse of the radius is the "curvature" of the curve at the origin.)

Hidden solution: [UNACCESSIBLE UUID '1TK'][UNACCESSIBLE UUID '1TM']

E24.5 Note:*Exercise 2, written exam 4 April 2009.* [1TN]

- Verify that for every $t > 1$ the equation

$$\sin x = x^t$$

  admits one and only one solution $x > 0$.
- Call $f(t)$ this solution, determine the image of the function t and show that it is strictly increasing and continuous on $(1, +\infty)$.
- Prove that f is extended by continuity to $t = 1$ and discuss the existence of the right derivative of the prolonged function at that point.

Hidden solution: [UNACCESSIBLE UUID '1TP']

E24.6 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous fuction such that $\cos(f(x))$ is differentiable: can it be deduced that f is differentiable? *If it is true, prove it. If it is not true, produce an example.* [1TS]

E24.7 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f > 0$ and $\log(f(x))$ is convex: can it be deduced that f is convex? *If it is true, prove it. If it is not true, produce an example.* [1TT]

E24.8 Let $f, g : \mathbb{R} \to \mathbb{R}$ be class C^∞ function, with $g > 0$: show that f/g is a class C^∞ function. [1TV]

E24.9 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $\rho > 0$, and let $f(0) = f'(0) = \ldots = f^{(n)}(0) = 0$; show that the function $g(x) = f(x)/x^n$ is extendable to $x = 0$; show that (the extension of) g coincides with an appropriate power series $g(x) = \sum_{n=0}^{\infty} b_n x^n$. What can be said about the radius of convergence of g? [1TW]

E24.10 Note:*Dirichlet criterion for integrals.* [1TX]

Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous, where f is positive and monotonic decreasing with $\lim_{x\to\infty} f(x) = 0$, while

$$\sup_{x>0} \left| \int_0^x g(t)\, \mathrm{d}t \right| < \infty \quad .$$

Then prove that

$$\lim_{x\to\infty} \int_0^x f(t) g(t)\, \mathrm{d}t$$

converges.

E24.11 Note:*written exam 12/1/2013.* [1TY]

Given a subset E of $\mathbb{N}$ and an integer $n \in \mathbb{N}$, the expression

$$\frac{\mathrm{card}(E \cap \{0, 1, \ldots, n\})}{n+1}$$

indicates which fraction of the segment $\{0, 1, \ldots, n\}$ is contained in E. The notion of "density" in $\mathbb{N}$ of E refers to the behavior of such fractions as n tends to infinity. Precisely, we define the upper density $\overline{d}(E)$ of E and its lower density $\underline{d}(E)$ as

$$\overline{d}(E) = \limsup_{n\to\infty} \frac{\mathrm{card}(E \cap \{0, 1, \ldots, n\})}{n+1} \quad ,$$

$$\underline{d}(E) = \liminf_{n\to\infty} \frac{\mathrm{card}(E \cap \{0, 1, \ldots, n\})}{n+1} \quad .$$

If $\overline{d}(E) = \underline{d}(E) = d \in [0, 1]$, E is said to have density d. (See also [62].)

1. Prove that, for every $\alpha \in \mathbb{R}, \alpha \geq 1$, the set $E_\alpha = [n\alpha] : n \in N$ has density $d = 1/\alpha$ (the symbol $[x]$ indicates the integer part of $x \in R$).
2. Let $E = \{m_0, m_1, \ldots, m_k, \ldots\}$ be an infinite set, with $m_0 < m_1 < \ldots < m_k < \ldots$. Prove that $\overline{d}(E) = \limsup_{k\to\infty} \frac{k}{m_k}$ and $\underline{d}(E) = \liminf_{k\to\infty} \frac{k}{m_k}$.
3. Find a set E with $\overline{d}(E) = \overline{d}(\mathbb{N} \setminus E) = 1$.

E24.12 Note:*exercise 6 in the written exam 13/1/2011.* [1TZ]

Each integer $n \geq 1$ decomposes uniquely as $n = 2^k d$, with $k \in N$ and d odd integer. Consider the sequence $a_n = d/2^k$ and compute

1. its upper and lower limit;
2. the set of limit points.

E24.13 Topics:matrix, determinant. Note:*exercise 4 in the pseudo-homework of 14/3/2013.* [1V0]

†120 Use the previous exercise 24.2!

1. Let $A \in \mathbb{R}^{2\times 2}$ be a 2 by 2 matrix. Identifying $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^4$, calculate the gradient of the determinant, and verify that it is nonzero if and only if the matrix is nonzero.
2. Let Z be the set of matrices $\mathbb{R}^{2\times 2}$ with zero determinant. Show that it is a closed set with an empty interior.

Hidden solution: [UNACCESSIBLE UUID '1V1']

E24.14 Topics:matrix,determinant.Difficulty:*. [1V2]

Prove Jacobi's formula:

$$\frac{d}{da_{i,j}} \det(A) = C_{i,j} \quad ,$$

where $a_{i,j}$ is the element of A in row i and column j, and C is the matrix of cofactors of A, which is the transpose of the adjoint matrix $\mathrm{adj}(A)$. Consequently, if $F : \mathbb{R} \to \mathbb{C}^{n\times n}$ is differentiable, then

$$\frac{d}{dt} \det F(t) = \mathrm{tr}\left(\mathrm{adj}(F(t))\, \frac{dF(t)}{dt}\right)$$

where $\mathrm{tr}(X)$ is the trace of X.

Hint: use Laplace's expansion for the determinant.

Hidden solution: [UNACCESSIBLE UUID '1V3']

E24.15 Topics:matrix,determinant.Prerequisites:24.14.Difficulty:*. [1V4]

We want to generalize the results of the previous exercise 24.13 to the case of matrices $n \times n$.

Recall the following properties of the determinant of matrices $A \in \mathbb{R}^{n\times n}$.

- The rank is the dimension of the image of A (considered as a linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$) and is also the maximum number of linearly independent columns in A.
- A has rank n if and only $\det(A) \neq 0$.
- If you exchange two columns in A, the determinant changes sign;
- if you add a multiple of another column to a column, the determinant does not change.
- The characterization of rank through minors, *"The rank of A is equal to the highest order of an invertible minor of A"*.
- Laplace's expansion of the determinant, and Jacobi's formula (*cf* 24.14).
- The determinant of A is equal to the determinant of the transpose; So every previous result holds, if you read "row" instead of "column".

See also in [65, 53].

Show the following results.

1. Show that the gradient of the function $\det(A)$ is not zero, if and only if the rank of A is at least $n-1$.
2. Let Z be the set of matrices $\mathbb{R}^{n\times n}$ with null determinant. Show that it is a closed set with an empty interior.

3. Fix B a matrix with rank at most $n-2$, show that the thesis of the theorem is false in the neighborhoods U_B of the matrix B, in the sense that $Z \cap U_B$ is not contained in a surface[†121].

Hidden solution: [UNACCESSIBLE UUID '1V6']

E24.16 Prove Young's inequality: fixed $a, b > 0$, $p, q > 1$ such that $1/p + 1/q = 1$ then [1V7]

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \tag{24.17}$$

with equality if and only if $a^p = b^q$; prove this using an appropriate function study.

Hidden solution: [UNACCESSIBLE UUID '1V8']

See also 15.d.3.

E24.18 Determine, among the triangles inscribed in the unit circle, the one of maximum area. [1Q6]

§24.a Functional equations

Exercises

E24.a.1 Note:*exercise 1, June 7th 2010.* [1V9]

Prove that there exists one and only one continuous function f on the interval $[-1, 1]$ such that

$$f(x) = 1 + \frac{x}{2} f\big(x^2\big) \quad \forall x \in [-1, 1] \quad .$$

Prove that f is representable as a power series centered at zero; and that the radius of convergence is one.

Hidden solution: [UNACCESSIBLE UUID '1VB']

E24.a.2 Difficulty:*.Note:*exercise 3, written exam, June 30th, 2017.* [1VC]

Consider the problem

$$\begin{cases} y'(x) = y(x^2) \\ y(0) = 1 \end{cases}$$

(this is not a Cauchy problem).

- Show that, for every $r < 1$, there is only one solution defined on $I = (-r, r)$, and deduce that the same is true for $r = 1$.
- Show that the solution is representable as the sum of a power series centered in 0 and converging on the interval $[-1, 1]$.

Hidden solution: [UNACCESSIBLE UUID '1VD']

E24.a.3 Note:*exercise 3, written exam, June 23th 2012.* [1VF]

Prove that there is one and only one continuous function f on interval $[0, 1]$ that satisfies the condition

$$f(x) = \sin(x) + \int_0^1 \frac{f(t)}{x^2 + t^2 + 1} \, dt \quad \forall x \in [0, 1] \quad .$$

E24.a.4 Note:*exercise 4, written exam, June 23th, 2012.* [1VG]

A function $f(x) = \sum_{n=0}^{\infty} a_n x_n$, analytic in a neighborhood of 0, satisfies on its domain the conditions

$$\begin{cases} f'(x) = 1 + f(-x) \\ f(0) = c \end{cases} ;$$

(note that this is not a Cauchy problem!).

- Determine f.
- Prove that the function found is the only solution, in the set of all functions that can be derived in a neighborhood of 0.

E24.a.5 [1VH]

- Show that there is an unique continuous function$f : (-1, 1) \to \mathbb{R}$ that satisfies

$$f(x) = x\cos(f(x)) \ .$$

- Fixed a, b, show that there exist a finite number of continuous $f : (-a, b) \to \mathbb{R}$ satisfying

$$f(x) = x\cos(f(x)) \ \forall x \in (a, b).$$

Hidden solution: *[UNACCESSIBLE UUID '1VJ']*

§24.b Vector Fields [1PW]

Exercises

E24.b.1 Note:*exercise 4, written exam 20 June 2017.* [1PX]

Let F be a continuous vector field on $\mathbb{R}^n \setminus \{0\}$, such that, for every $x \neq 0$, $F(x)$ is a scalar multiple of x. For $r > 0$, we denote with S_r the sphere of radius r centered in 0.

- Prove that, for each regular arc γ with support contained in a sphere S_r , we have $\int_\gamma F = 0$.
- Prove that, if such a field F is conservative, then $|F(x)|$ is constant on every sphere S_r, and therefore that $F(x) = x\rho(|x|)$ with $\rho : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ continuous.

Hidden solution: *[UNACCESSIBLE UUID '1PY']*

†121 This problem is simpler than you think... There are too many matrices with zero determinant close to B...

UUID

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