Exercises

E3.i.11 A well-formed formula in propositional logic is a **tautology** if for each *evaluation* the formula is always true. Suppose A, B, C are well-formed formulas. Show that the following properties of connectives are tautologies. [†9] [OON]

$$A \Rightarrow A \quad \text{identity law}$$
$$\neg(\neg A) \Leftrightarrow A \quad \text{law of double negation}$$
$$A \vee A \Leftrightarrow A \;\;,\;\; A \wedge A \Leftrightarrow A \quad \text{laws of idempotence}$$
$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A) \quad \text{law of opposition, or of the contrapositive [†10]} \tag{3.i.12}$$
$$(A \Rightarrow B) \Leftrightarrow (\neg A \vee B) \Leftrightarrow (\neg(A \wedge \neg B)) \quad \text{equivalence of implication, conjunction and disjunction} \tag{3.i.13}$$
$$A \wedge B \Leftrightarrow \neg(\neg A \vee \neg B) \quad \text{first law of De Morgan} \tag{3.i.14}$$
$$A \vee B \Leftrightarrow \neg(\neg A \wedge \neg B) \quad \text{second law of De Morgan} \tag{3.i.15}$$
$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C) \quad \text{distributive property of the conjunction with respect to the disjunction} \tag{3.i.16}$$
$$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C) \quad \text{distributive property of the disjunction with respect to the conjunction} \tag{3.i.17}$$
$$A \wedge B \Leftrightarrow B \wedge A \quad \text{commutative property of } \wedge$$
$$A \vee B \Leftrightarrow B \vee A \quad \text{commutative property of } \vee$$
$$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C \quad \text{associative property of } \wedge$$
$$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C \quad \text{associative property of} \vee \tag{3.i.18}$$

These last two properties allow to omit parentheses in sequences of conjunctions or disjunctions.

The property (3.i.13),(3.i.14),(3.i.15) they say that we could base all logic on connectives alone $\neg$ and $\wedge$, (or on $\neg$, $\vee$).

[†9] These lists are taken from Section 1.3 in [14], or [32].

[†10] The clause $(\neg B \Rightarrow \neg A)$ is called *"contrapositive"* of $(A \Rightarrow B)$.

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Other important tautologies, often used in logical reasoning.

$A \vee \neg A$	excluded middle	
$\neg(A \wedge \neg A)$	law of non-contradiction	
$(A \wedge (A \Rightarrow B)) \Rightarrow B$	modus ponens	(3.i.19)
$(\neg B \wedge (A \Rightarrow B)) \Rightarrow \neg A$	modus tollens	(3.i.20)
$\neg A \Rightarrow (A \Rightarrow B)$	negation of the antecedent	
$B \Rightarrow (A \Rightarrow B)$	affirmation of the consequent	
$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)$	exporting	
$((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$	proof by parts	
$((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$	proof by cases	
$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$	hypothetical syllogism, or transitivity of implication	
$(A \vee (A \wedge B)) \Leftrightarrow A \wedge (A \vee B) \Leftrightarrow$		
$(A \vee F) \Leftrightarrow (A \wedge V) \Leftrightarrow A$	absorption laws	
$F \Rightarrow B$	first law of Pseudo Scotus, or *ex falso sequitur quodlibet*	
$A \Rightarrow (\neg A \Rightarrow B)$	second law of Pseudo Scoto	
$(\neg A \Rightarrow F) \Leftrightarrow A$	proof by contradiction	
$((A \wedge \neg B) \Rightarrow F) \Leftrightarrow (A \Rightarrow B)$	proof by contradiction, with hypothesis and thesis	
$(\neg A \Rightarrow A) \Rightarrow A$	consequentia mirabilis	(3.i.21)

[[OOP]]