Exercises

3.147 [01P] *(Solved on 2022-11-15)* Let D, C be non-empty sets. A **partial function** from D in C is a function $\varphi : B \to C$ where $B \subseteq D$. (The definition of "*function*" is in [1Y6]).

It can be convenient to think of the partial function as a relation $\Phi \subseteq D \times C$ such that, if $(x, a), (x, b) \in \Phi$ then $a = b$ (see [23X]). The two notions are equivalent in this sense: given Φ we build the domain of φ, which we will call B, with the projection of Φ on the first factor i.e. $B = \{x \in D : \exists c \in C, (x, c) \in \Phi\}$, and we define $\varphi(x) = c$ as the only element $c \in C$ such that $(x, c) \in \Phi$; vice versa Φ is the graph of φ.

Partial functions, seen as relations Φ, are of course sorted by inclusion; equivalently $\varphi \leq \psi$ if $\varphi : B \to C$ and $\psi : E \to C$ and $B \subseteq E \subseteq D$ and $\varphi = \psi_{|B}$.

Let now U be a **chain**, i.e. family of partial functions that is totally ordered according to the order previously given; seeing each partial function as a relation, let Ψ be the union of all relations in U; show that Ψ is the graph of a partial function $\psi : E \to C$, whose domain E is the union of all the domains of the functions in U, and whose image I is the union of all images of functions in U

If moreover all functions in U are injective, show that ψ is injective.

Solution 1. *[01Q]*