Exercises

E3.ii.47 [02D] Let V be a real vector space. Let $B \subseteq V$ be a subset. A **finite linear combination** v of elements of B is equivalently defined as

- $v = \sum_{i=1}^{n} \ell_i b_i$ where $n = n(v) \in \mathbb{N}$, $\ell_1, \dots, \ell_n \in \mathbb{R}$ and $b_1, \dots, b_n$ are elements of B;
- $v = \sum_{b \in B} \lambda(b) b$ where $\lambda : B \to \mathbb{R}$ but also $\lambda(b) \neq 0$ only for a finite number of $b \in B$.

We call $\Lambda \subseteq \mathbb{R}^B$ the set of functions λ as above, which are non-null only for a finite number of arguments; Λ is a vector space: so the second definition is less intuitive but is easier to handle.

We will say that B **generates** (or, **spans**) V if every $v \in V$ is written as finite linear combination of elements of B.

We will say that the vectors of B are **linearly independent** if $0 = \sum_{b \in B} \lambda(b) b$ implies $\lambda \equiv 0$; or equivalently that, given $n \geq 1$, $\ell_1, \dots, \ell_n \in \mathbb{R}$ and $b_1, \dots, b_n \in B$ all different, the relation $\sum_{i=1}^{n} \ell_i b_i = 0$ implies $\forall i \leq n, \ell_i = 0$.
We will say that B is an **algebraic basis** (also known as **Hamel basis**) if both properties apply.

If B is a basis then the linear combination that generates v is unique (i.e. there is only one function $\lambda \in \Lambda$ such that $v = \sum_{b \in B} \lambda(b) b$).

Show that any vector space has an *algebraic basis*. Show more in general that for each $A, G \subseteq V$, with A family of linearly independent vectors and G generators, there is an *algebraic basis* B with $A \subseteq B \subseteq G$.

Solution 1. *[02G]*

The proof in general requires Zorn's Lemma; indeed this statement is equivalent to the Axiom of Choice; this was proved by A. Blass in [8]; see also Part 1 §6 [24].