Exercise 7.14. [02F]Prerequisites:[1Z7].*Let a_n, b_n be real sequences (which can have variable signs, take value zero, and are not necessarily infinitesimal); let $X = \mathbb{R}^{\mathbb{N}}$ the space of all sequences.*
Recall that the notation $a_n = O(b_n)$ means:

$$\exists M > 0,\ \exists \overline{n} \in \mathbb{N},\ \forall n \in \mathbb{N}, n \geq \overline{n} \Rightarrow |a_n| \leq M|b_n|\ .$$

Show these results:

- *for $a, b \in X, a = (a_n)_n, b = (b_n)_n$ consider the relation*

  $$aRb \iff a_n = O(b_n)$$

  prove that R is a preorder;

- *define $x \asymp y \iff (xRy \wedge yRx)$ then $\asymp$ is an equivalence relation, R is invariant for $\asymp$, and the projection $\preceq$ is an order relation on $X/\asymp$ (hint: use the Prop. [1Z7]).*

- *Define (as usually done)*

  $$\hat{a} \prec \hat{b} \iff (\hat{a} \preceq \hat{b} \wedge \hat{a} \neq \hat{b})$$

  for $\hat{a}, \hat{b} \in X/\asymp, (a_n)_n \in \hat{a}, (b_n)_n \in \hat{b}$ representatives; assuming $b_n \neq 0$ (eventually in n), prove that

  $$\hat{a} \prec \hat{b} \iff 0 = \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} < \infty \quad .$$

The above discussion is related to Definition 3.2.3 (and following) in [3].