Exercises

E3.88 [02M]Difficulty:*.[a] Consider the following quotient of the family of all integer valued sequences

$$\mathbb{X} = \{a : \mathbb{N} \to \mathbb{N}\}/\sim$$

where we define $a \sim b$ iff $a_k = b_k$ eventually in k.

We define the ordering

$$a \preceq b \iff \exists n \text{ s.t. } \forall k \geq n, a_k \leq b_k$$

that is, $a \preceq b$ when $a_k \leq b_k$ eventually. This is a preorder and

$$a \sim b \iff (a \preceq b \wedge b \preceq a)$$

so it passes to the quotient were it becomes an ordering, see Prop. [1Z7].

Let a^k be an increasing sequence of sequences, that is, $a^k \preceq a^{k+1}$; we readily see that it has an upper bound b, by defining

$$b_n = \sup_{h,k \leq n} a^k_h .$$

We can then apply the Zorn Lemma to assert that in the ordered set $(\mathbb{X}, \preceq)$ there exist maximal elements.

Given a, b we define

$$a \vee b = (a_n \vee b_n)_n$$

then it is easily verified that $a \preceq a \vee b$. So this a *direct* ordering, see [06N].

We conclude that the ordered set $(\mathbb{X}, \preceq)$ has an unique maximum, by [06S].

This is though false, since for any sequence a the sequence $(a_n + 1)_n$ is larger than that.

What is the mistake in the above reasoning? What do you conclude about $(\mathbb{X}, \preceq)$?

[a] Originally published in https://dida.sns.it/dida2/Members/mennucci/curiosa/