Exercises

E0.1 [06G] Let $R \subseteq X \times X$ be a relation; taken $x, y \in X$ we will write xRy instead of $(x, y) \in R$. Suppose R is *reflexive* i.e. xRx for every x, and that it is *transitive* that is for every $x, y, z \in X$ you have $xRy \wedge yRz \Rightarrow xRz$.

R is not necessarily an order relation, because we have not assumed that it is *antisymmetric*: however, we can ”build” an order relation ”identifying with each other” the elements of X on which R is not antisymmetric. Let's see the construction in detail.

(a) Let now $\sim$ the relation given by

$$x \sim y \iff xRy \wedge yRx$$

show that it is an equivalence relation.

(b) Consider the quotient $Y = X/\sim$, we want to show how R passes to the quotient and produces a relation T on Y. Show that if $x, y, \tilde{x}, \tilde{y} \in X$ and we have $x \sim \tilde{x}, y \sim \tilde{y}$ then $xRy \iff \tilde{x}R\tilde{y}$. So let's define T as the set of all pairs of classes of equivalence whose products are contained in R i.e. $T = \{(z, w) \in Y^2 : z \times w \subseteq R\}$; this can also be written as

$$zTw \iff (\forall x \in z, \forall y \in w, xRy)$$

we abbreviate this definition as $[x]T[y] \iff xRy$ (which is a good definition because the right member does not depend on the choice of representatives in the classes).

(c) Finally, show that T is an order relation on Y.

Solution 1. [06H]