Proof. [090] Trace of proof. Note that the proof only uses Peano's axioms and induction. Given $m \in \mathbb{N}, m \neq 0$ we recall that $S^{-1}(m)$ is the predecessor, see [1YP] (using the arithmetic we may write

$$S^{-1}(m) = m - 1 \quad , \quad S(k) = k + 1$$

but this theorem is needed to define the arithmetic...) For any given $R \subseteq \mathbb{N} \times A$ we define the projection on the first component

$$\pi(R) = \{n \in \mathbb{N}, \exists x \in A, (n, x) \in R\} \, .$$

Consider the family $\mathcal{F}$ of relations $R \subseteq \mathbb{N} \times A$ that satisfy

$$(0, a) \in R \qquad (*)$$

$$\forall n \geq 0, \forall y \in A, \quad (n, y) \in R \Rightarrow (S(n), g_n(y)) \in R \qquad (**)$$

We show that under these conditions $\pi(R) = \mathbb{N}$; we know that $0 \in \pi(R)$; if $m \in \pi(R)$, then there exists $x \in A$ for which $(m, x) \in R$ from which for ** follows $(S(m), g_m(x)) \in R$, and we obtain $S(m) \in \pi(R)$.
The family $\mathcal{F}$ is not empty because $\mathbb{N} \times A \in \mathcal{F}$. Let then T be the intersection of all relations in $\mathcal{F}$. T is therefore the least relation in $\mathcal{F}$.
It is possible to verify that T satisfies the previous * and ** properties. In particular $\pi(T) = \mathbb{N}$.
We must now show that T is the graph of a function (which is the desired f function), that is, that for every n there is a single $x \in A$ for which $(n, x) \in T$.
Let $A_n = \{x \in A, (n, x) \in T\}$; we write $|A_n|$ to denote the number of elements in A_n; since $\pi(T) = \mathbb{N}$ then $|A_n| \geq 1$ for every n. We will show that $|A_n| = 1$ for each n. We will prove it by induction. Let

$$P(n) \doteq |A_n| = 1 \quad .$$

Let's see the induction step.
Suppose by contradiction that $|A_m| = 1$ but $|A_{Sm}| \geq 2$; morally at m the graph of the function f "forks" and the function becomes "multivalued". We define for convenience $w = g_m(x), k = Sm$; we may remove some elements to T (those that do not have a "predecessor" according to the relation **) defining

$$\tilde{T} = T \setminus \{(k, y) \, : \, y \in A, y \neq w\}$$

it is possible to show that $\tilde{T}$ satisfies * and **, but $\tilde{T}$ would be smaller than T, against the minimality of T. To prove that $P(0)$ holds, we define $k = 0, w = a$ and proceed in the same way.
The previous reasoning also shows that the function is unique, because if the graph G of any function satisfying to * and ** must contain T, then $T = G$. □