Exercises

E7.3 [0CX] Difficulty:*.

Let $a_{n,m}$ be a real valued sequence [a] with two indexes $n, m \in \mathbb{N}$. Suppose that

- for every m the limit $\lim_{n\to\infty} a_{n,m}$ exists, and that
- the limit $\lim_{m\to\infty} a_{n,m} = b_n$ exists uniformly in n and is finite, *i.e.*

$$\forall \varepsilon > 0,\ \exists m \in \mathbb{N}\ \forall n \in \mathbb{N},\ \forall h \ge m\ \ |a_{n,h} - b_n| < \varepsilon\ \ .$$

then

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m} \tag{7.3}$$

in the sense that if one of the two limits exists (possibly infinite), then the other also exists, and they are equal.

Find a simple example where the two limits in (7.3) are infinite.

Find an example where $\lim_{m\to\infty} a_{n,m} = b_n$ but the limit is not uniform and the previous equality (7.3) does not apply.

Solution 1. *[0CZ]*

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[a] This result applies more generally when $a_{n,m}$ are elements of a metric space; moreover a similar result occurs when the limits $n \to \infty$ and/or $m \to \infty$ are replaced with limits $x \to \hat{x}$ and/or $y \to \hat{y}$ where the above variables move in metric spaces. See for example [1JS].