E7.16 [ODN] Note:*Babylonian method for square root.* Let $S > 0$ and consider the sequence defined by recurrence as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right) \quad ;$$

show that $x_n \to \sqrt{S}$ and that, for $S \in [1/4, 1]$ and $x_0 = 1$, convergence is superquadratic, i.e.

$$\left|x_n - \sqrt{S}\right| \le 2^{1-2^n} \quad .$$

Find a function $f(x)$ (dependent on S) such that the previous iteration can be seen as a Newton's method, i.e.

$$x - \frac{f(x)}{f'(x)} = \frac{1}{2}\left(x + \frac{S}{x}\right).$$

Generalize the Babylonian method to find a root $\sqrt[k]{S}$.

Solution 1. [ODP]