Exercises

E7.31 [0F8] A sequence is given $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$: prove that for every $l \in \mathbb{R}$ there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{1,-1\}$ for each n, such that

$$\sum_{n=0}^{\infty}(\varepsilon_n a_n) = l \quad .$$

If instead $\sum_{n=0}^{\infty} a_n = S < \infty$, what can be said about the set E of the sums $\sum_{n=0}^{\infty}(\varepsilon_n a_n) = l$, for all possible choices of $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{1,-1\}$ for every n?

- Analyze cases where $a_n = 2^{-n}$ or $a_n = 3^{-n}$
- Show that E is always closed.
- Under what assumptions do you have that $E = [-S, S]$?

Hint. Let $\tilde{E}$ be the set of sums $\sum_n(\varepsilon_n a_n) = l$, to vary by $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{0,1\}$ for each n; note that $\tilde{E} = \{(S+x)/2 \,:\, x \in E\}$.