Definition 7.51. [0FR] Prerequisites:[06V], [06Y], Sec. [1YY].
Given J a (possibly partially) ordered and filtering set, and given $f : J \to \mathbb{R}$, we want to define the concept of limit of $f(j)$ "for $j \to \infty$". [a].

- *We will say that*

$$\lim_{j\in J} f(j) = l \in \mathbb{R}$$

if

$$\forall \varepsilon > 0\, \exists k \in J\, \forall j \in J,\ j \geq k \Rightarrow |l - f(j)| < \varepsilon \quad .$$

Similarly limits are defined $l = \pm\infty$ (imitating the definitions used when $J = \mathbb{N}$.) (This is the definition in the course notes, chap. 4 sect. 2 in [3])

- *Equivalently we can say that*

$$\lim_{j\in J} f(j) = l \in \overline{\mathbb{R}}$$

if for every neighborhood U of l we have that $f(j) \in U$ eventually for $j \in J$; where eventually *has been defined in* [06Y].

- *We recall from* [231] *that* "a neighborhood of ∞ in J" *is a subset $U \subseteq J$ such that $\exists k \in J \forall j \in J, j \geq k \Rightarrow j \in U$. Then we can imitate the definition* [20D].

Fixed $l \in \overline{\mathbb{R}}$ we have $\lim_{j\in J} f(j) = l$ when for every "full" neighborhood V of l in $\mathbb{R}$, there exists a neighborhood U of ∞ in J such that $f(U) \subseteq V$.

In particular, this last definition can be used to define the limits of $f : J \to E$ where E is a topological space.

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[a] Note that ∞ is a symbol but it is not an element of J : if it were it should be the maximum, but a filtering set cannot have maximum, *cf* [06V]