Exercises

E8.16 [0GQ] Prerequisites: [06N], [06M], [06V]. Difficulty: *. *(Replaces 29W)* Let (X, τ) be a topological space. Consider the descending ordering between sets [a], with this ordering τ is a directed set; we note that it has minimum, given by $\emptyset$.

Now suppose the topology is Hausdorff. Then taken $x \in A$, let $\mathcal{U} = \{A \in \tau : x \in A\}$ be the family of the open sets that contain x: show that $\mathcal{U}$ is a directed set; show that it has minimum if and only if the singleton $\{x\}$ is open (and in this case the minimum is $\{x\}$). [b]

Solution 1. *[0GR]*

By the exercise [06V], when $\{x\}$ is not open then $\mathcal{U}$ is a filtering set, and therefore can be used as a family of indices to define a nontrivial "limit" (see Remark [237]). We will see applications in section [2B8].

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[a] To formally reconnect to the definition seen in [06N] we define $A \preceq B \iff A \supseteq B$ and associate the ordering $\preceq$ with τ.

[b] Note that, the singleton $\{x\}$ is open iff x is an isolated point.