Exercises

E8.j.2 [0MM][a] Prerequisites:[2F7],[0KK].Difficulty:*.Let Ω be a non-empty set; let's consider $X = \mathbb{R}^\Omega$.

1. Let
$$U^f_{E,\rho} = \{g \in X, \forall x \in E, |f(x) - g(x)| < \rho\}$$
where $f \in X$, $\rho > 0$ and $E \subset \Omega$ is finite. Show that the family of these $U^f_{E,\rho}$ satisfies the requirements of [0KZ], and is therefore a *base* for a topology τ (Hint: use [2F7]). This topology is the *product topology* of topologies of $\mathbb{R}$.

   In particular for each $f \in X$ the sets $U^f_{E,\rho}$ are a fundamental system of neighborhoods.
2. Check that the topology is T_2.
3. Note that X is a vector space, and show that the "sum" operation is continuous, as an operation $X \times X \to X$; to this end, show that if $f, g \in X, h = f + g$, for every neighborhood V_h of h there are neighborhoods V_f, V_g of f, g such that $V_f + V_g \subseteq V_h$.
4. Given $B_i \subset \mathbb{R}$ open and non-empty, one for each $i \in \Omega$, show that $\prod_i B_i$ is open if and only if $B_i = \mathbb{R}$ except at most finitely many i.

Solution 1. [0MN]

[a] These two exercises [0MM],[2BP], are taken from a text originally published by Prof. Ricci in http://dida.sns.it/dida2/cl/08-09/folde0/pdf9 in March 2014.