Exercises

E10.53 [0QM] Let's review the exercise [0PT].

Having fixed $(X_1, d_1), \dots, (X_n, d_n)$ metric spaces, let $X = X_1 \times X_1 \times \cdots \times X_n$.

Let d be the distance

$$d(x,y) = \max_{i=1,\dots n} d_i(x_i, y_i) \ .$$

This is the same d defined as in eqn. [(9.26)] inside [0PT], setting $\varphi(x) = \max_{i=1\dots n} |x_i|$. We indicate with $B^d(x,r)$ the ball in (X,d) of center $x \in X$ and radius $r > 0$.

We want to show that d induces the product topology on X, using the results seen in Sec. [2B5].

Taken $t \in X_i, r > 0$ we indicate with $B^{d_i}(t,r)$ the ball in metric space (X_i, d_i). Let $\mathcal{B}_i$ be the family of all balls in (X_i, d_i).

Let $\mathcal{B}$ be defined as

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} B^{d_i}(x_i, r_i) \ : \ \forall i, x_i \in X_i, r_i > 0 \right\}$$

This is the same $\mathcal{B}$ defined in [0M5].

Show that every ball $B^d(x,r)$ in (X,d) is the Cartesian product of balls $B^{d_i}(x_i, r)$ in (X_i, d_i). So let $\mathcal{P}$ be the family of balls $B^d(x,r)$ in (X,d).

From [0QJ] we know that $\mathcal{P}$ is a base for the standard topology in the metric space (X,d).

Use [0M7] to show that $\mathcal{P}$ and $\mathcal{B}$ generate the same topology τ.

Use [0M5] to prove that τ is the product topology.

We conclude that the distance d generates the product topology.