Exercises

E9.58 [0R5] Let (X,d) be a metric space where X is also a group. Let Θ be a subgroup.

We define that $x \sim y \iff xy^{-1} \in \Theta$. It is easy to verify that this is an equivalence relation. Let $Y = X/\sim$ be the quotient space. [a]

Suppose d is invariant with respect to left multiplication by elements of Θ:

$$d(x,y) = d(\theta x, \theta y) \ \ \forall x, y \in X, \forall \theta \in \Theta \ . \qquad (9.58)$$

(This is equivalent to saying that, for every fixed $\theta \in \Theta$ the map $x \mapsto \theta x$ is an isometry). We define the function $\delta : Y^2 \to \mathbb{R}$ as in [(9.57)].

- Show that, taken $s, t \in X$,

  $$\delta([s],[t]) = \inf\{d(s,\theta t) : \theta \in \Theta\} \qquad (9.59)$$

  where $[s]$ is the class of elements equivalent to s.
- Show that $\delta \geq 0$, that δ is symmetric and that δ satisfies the triangle inequality.
- Suppose that, for every fixed $t \in X$, the map $\theta \mapsto \theta t$ is continuous from Θ to X; suppose also that Θ is closed: then δ is a distance. [b]

Solution 1. *[0R6]*

---

[a] If Θ is a normal subgroup then $Y = X/\sim$ is also written as $Y = X/\Theta$, and this is a group.

[b] Note that, using [161], under these hypotheses the map of multiplication $(\theta, x) \mapsto \theta x$ is continuous from $\Theta \times X$ to X.