Exercises

E13.16 [156]Prerequisites:[155].Let $f : X_1 \to X_2$ with (X_1, d_1) and (X_2, d_2) metric spaces.

A monotonic (weakly) increasing function $\omega : [0,\infty) \to [0,\infty]$, with $\omega(0) = 0$ and $\lim_{t\to 0+} \omega(t) = 0$, such that

$$\forall x, y \in X_1, \;\; d_2(f(x), f(y)) \le \omega(d_1(x,y)) \,, \tag{13.17}$$

is called **continuity modulus** for the function f. (Note that f can have many continuity moduli).

For example, if the function f is Lipschitz, i.e. there exists $L > 0$ such that

$$\forall x, y \in X_1, \;\; d_2(f(x), f(y)) \le L\, d_1(x,y)$$

then f satisfies the eqz. (13.17) by placing $\omega(t) = Lt$.

We will now see that the existence of a continuity modulus is equivalent to the uniform continuity of f.

- If f is uniformly continuous, show that the function

  $$\omega_f(t) = \sup\{d_2(f(x), f(y)) \;:\; x, y \in X_1, d_1(x,y) \le t\} \tag{13.18}$$

  is the smallest continuity modulus.[a]
- Note that the modulus defined in (13.18) may not be continuous, and may be infinite for t large — find examples of this behaviour.
- Also show that if f is uniformly continuous, there is a modulus that is continuous where it is finite.
- Conversely, it is easy to verify that if f has a continuity modulus, then it is uniformly continuous.

If you don't know metric space theory, you can prove the previous results in case $f : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$. (See also the exercise [15W], which shows that in this case the modulus ω defined in (13.18) is continuous and is finite).

Solution 1. *[157]*

[[15B]]

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[a] Note that the family on which the upper bound is calculated always contains the cases $x = y$, therefore $\omega(t) \ge 0$.