15 Convex functions and sets [16V]

We will now discuss convexity. For simplicity, all results are presented using $\mathbb{R}^n$ as domain; but most results hold more in general in a generic vector space.

15.1 Convex sets

[2F0]

Topology

Exercises

E15.1.1 [16Y]

E15.1.2 [16Z]

E15.1.3 [170]

E15.1.4 [172]

E15.1.5 [174]

E15.1.6 [176]

E15.1.7 [178]

E15.1.8 [17B]

E15.1.9 [059]

Definition 15.1.10. *[2G4]*

See also exercises [122], [130] and [132].

15.1.1 Projection, separation

Exercises

E15.1.11 [17D]

E15.1.12 [17H]

E15.1.13 [17J]

E15.1.14 [17M]

E15.1.15 [17P]

E15.1.16 [17R]

E15.1.17 [17T]

E15.1.18 [17W]

15.2 Convex function

Definition 15.2.1. [17Y]

Definition 15.2.2. [17Z]

Convex functions enjoy a lot of interesting properties, this one below is just a small list.

... equivalent definitions

Exercises

E15.2.3 [180]

E15.2.4 [181]

Properties

The following is a list of properties for convex functions $f : C \to \mathbb{R}$ with $C \subseteq \mathbb{R}^n$. Obviously these properties also apply when $n = 1$; but when $n = 1$ proofs are usually easier, see the next section.

Exercises

E15.2.5 [182]

E15.2.6 [183]

E15.2.7 [184]

E15.2.8 [186]

E15.2.9 [188]

E15.2.10 [18B]

E15.2.11 [18C]

15.3 Real case

Let $I \subset \mathbb{R}$, then I is convex if and only it is an interval (see [0S0]). In the following we will consider $f : I \to \mathbb{R}$ where $I = (a, b)$ is an open interval.

Exercises

E15.3.1 [18F]

E15.3.2 [18H]

E15.3.3 [18J]

E15.3.4 [18K]

15.3.1 Convexity and derivatives

Exercises

E15.3.5 [18M]

E15.3.6 [18P]

E15.3.7 [18R]

E15.3.8 [18T]

E15.3.9 [18W]

See also the exercise [1BF] for the relationship between integral and convexity.

15.3.2 Convex functions with extended values

We consider convex functions that can also take on value $+\infty$. Let I be an interval.

Exercises

E15.3.10 [18Y]

E15.3.11 [18Z]

15.4 Additional properties and exercises

Exercises

E15.4.1 [191]

E15.4.2 [192]

E15.4.3 [194]

E15.4.4 [196]

See also exercise [1C3].

15.4.1 Distance function

Exercises

E15.4.5 [198]

E15.4.6 [19B]

15.4.2 Strictly convex functions and sets

Exercises

E15.4.7 [19C]

Remark 15.4.8. *[23N]*

One wonders now, what if f is strictly convex?

Definition 15.4.9. *[19D]*

(Note that a strictly convex set necessarily has a non-empty interior).

Remark 15.4.10. *[19F]*

Exercises

E15.4.11 [19G]

[[19J]]