Exercises

E14.17 [17J] Topics:separation. Difficulty:*.

This result applies in very general contexts, and is a consequence of Hahn–Banach theorem (which makes use of Zorn's Lemma); if $A \subset \mathbb{R}^n$ it can be proven in an elementary way, I invite you to try.

Given $A \subset \mathbb{R}^n$ open convex non-empty and $z \notin A$, show that there is a hyperplane P separating z from A, that is, $z \in P$ while A is entirely contained in one of the two closed half-spaces bounded by the hyperplane P. Equivalently, in analytical form, there exist $a \in \mathbb{R}, v \in \mathbb{R}^n, v \neq 0$ such that $\langle z, v\rangle = a$ but $\forall x \in A, \langle x, v\rangle < a$; and

$$P = \{y \in \mathbb{R}^n \ : \langle y, v\rangle = a\} .$$

The hyperplane P thus defined is called *supporting hyperplane* of z for A.

There are (at least) two possible proofs. A possible proof is made by induction on n; we can assume without loss of generality that $z = e_1 = (1, 0 \ldots 0), 0 \in A, a = 1$; keep in mind that the intersection of a convex open sets with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ is an open convex set in $\mathbb{R}^{n-1}$; this proof is complex but does not use any prerequisite. A second proof uses [176] and [17H] if $z \notin \partial A$; if $z \in \partial A$ it also uses [178] to find $(z_n) \subset (A^c)^\circ$ with $z_n \to z$.

Solution 1. [17K]