## Exercises

E15.c.6 [18M] Prerequisites: [18F]. Let $f : (a, b) \to \mathbb{R}$ be convex.

1. Show that, at every point, right derivative $d^+(x)$ and left derivative $d^-(x)$ exist (In particular $f$ is continuous).
2. Show that $d^-(x) \le d^+(x)$,
3. while, for $x < y$, $d^+(x) \le R(x, y) \le d^-(y)$.
4. hence $d^+(x)$ and $d^-(x)$ are monotonic weakly increasing.
5. Show that $d^+(x)$ is right continuous, while $d^-(x)$ is left continuous.
6. Also show that $\lim_{s\to x-} d^+(s) = d^-(x)$, while $\lim_{s\to x+} d^-(s) = d^+(x)$. In particular $d^+$ is continuous in $x$, if and only if $d^-$ is continuous in $x$, if and only if $d^-(x) = d^+(x)$.

   So $d^+, d^-$ are, so to speak, the same monotonic function, with the exception that, at any point of discontinuity, $d^+$ assumes the value of the right limit while $d^-$ the value of the left limit.
7. Use the above to show that $f$ is differentiable in $x$ if and only if $d^+$ is continuous in $x$, if and only if $d^-$ is continuous in $x$.
8. Eventually, prove that $f$ is differentiable, except in a countable number of points.

**Solution 1.** *[18N]*