Exercises

E16.4 [19S] Prerequisites:[19Q], [1HS].

Let $I \subset \mathbb{R}$ be an interval with extremes a, b. Let $f, f_n : I \to \mathbb{R}$ be continuous non-negative functions such that $f_n(x) \nearrow_n f$ pointwise (i.e. for every x and n we have $0 \le f_n(x) \le f_{n+1}(x)$ and $\lim_n f_n(x) = f(x)$). Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x \quad .$$

(Note if the interval is open or semiopen or unbounded then the Riemann integrals are understood in a generalized sense; in this case the right term can also be $+\infty$).

Solution 1. *[19T]*

The previous result is called *Monotonic Convergence Theorem* and holds in very general hypotheses; in the case of Riemann integrals, however, it can be seen as a consequence of the results [19Q] and [1HS].