16.48 [1GW] Let E be the set of horizontal lines

$$E = \{(x,0) : x \in \mathbb{R}\} \cup \bigcup_{n=1}^{\infty} \{(x, 1/n) : x \in \mathbb{R}\} \quad .$$

Find a function $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,y)$ class C^1 such that $E = \{(x,y) : f(x,y) = 0\}$.

Prove that necessarily $\partial_y f(0,0) = 0$.

Set $(\overline{x}, \overline{y}) = (0,0)$. Note that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $f(x, g(x)) = 0$! In fact, the function $g \equiv 0$ is the only function with such characteristics. Thus part of the thesis in the implicit function theorem is satisfied.

So explain precisely why the thesis of the implicit function theorem is not satisfied.