Exercises

E18.10 [1JG] We wonder if the previous classes $\mathcal{F}$ enjoy a *"rigidity property"*, that is, if from a more "weak" convergence in the class follows a more "strong" convergence. Prove the following propositions.

1. Let $f_n, f : I \to \mathbb{R}$ be continuous and monotonic (weakly) increasing functions, defined over a closed and bounded interval $I = [a, b]$. Suppose there is a dense set J in I with $a, b \in J$, such that $\forall x \in J, f_n(x) \to_n f(x)$, then $f_n \to_n f$ uniformly.

   Solution 1. *[1JH]*

2. Let $A \subseteq \mathbb{R}$ be open interval. Let $f_n, f : A \to \mathbb{R}$ be convex functions on A. If there is a set J dense in A such that $\forall x \in J, f_n(x) \to_n f(x)$, then, for every $[a, b] \subset A$, we have that $f_n \to_n f$ uniformly on $[a, b]$.

   Solution 2. *[1JJ]*

3. Let $f_n : I \to \mathbb{R}$ be a family of equicontinuous functions, [a] defined on a closed and bounded interval $I = [a, b]$, and let ω be their modulus of continuity. If there is a set J dense in $[a, b]$ such that $\forall x \in J, f_n(x) \to_n f(x)$, then, f extends from J to I so that it is continuous (with modulus ω), and $f_n \to_n f$ uniformly on $[a, b]$.

   Solution 3. *[1JK]*

4. Let $f_n, f : I \to \mathbb{R}$ be polynomials of degree less than or equal to N, seen as functions defined on an interval $I = [a, b]$ closed and bounded; fix $N + 1$ distinct points $a \le x_0 < x_1 < x_2 < \ldots < x_N \le b$; assume that, for each x_i, $f_n(x_i) \to_n f(x_i)$: then f_n converge to f uniformly, and so do each of their derivatives $D^k f_n \to_n D^k f$ uniformly.

   Solution 4. *[1JM]*

Also look for counterexamples for similar propositions, when applied to the other classes of functions seen in the previous exercise.

[a] Definition is in [1HR]