Exercises

18.12 [1M3]Prerequisites: [1K9], [1KQ], [20V], [20W]. It is customary to define

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

for $z \in \mathbb{C}$. We want to reflect on this definition.

- First, for each $z \in \mathbb{C}$, we can actually define

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

  (Note that the radius of convergence is infinite — as it easily occurs using the root criterion [219]).

- We note that $f(0) = 1$; we define $e = f(1)$ which is *Euler's number*[a]
- Show that $f(z + w) = f(z)f(w)$ for $z, w \in \mathbb{C}$.
- It is easy to verify that $f(x)$ is monotonic increasing for $x \in (0, \infty)$; by the previous relation, $f(x)$ is monotonic increasing for $x \in \mathbb{R}$.
- Then show that, for $n, m > 0$ integer, $f(n/m) = e^{n/m}$ (for the definition of $e^{n/m}$ see [20V]).
- Deduce that, for every $x \in \mathbb{R}$, $f(x) = e^x$ (for the definition of e^x see [20W])

Solution 1. *[1M4]*

[a] Known as *numero di Nepero* in Italy.