Exercises

18.24 [1N0]Prerequisites:[1T1].Difficulty:**.In the general case (even when we do not know if A, B commute), we can express $\exp(A + sB)$ using a power series. Define

$$C(t) = \exp(-tA) B \exp(tA)$$

and (recursively) set $Q_0 = \mathbb{I}$ (the identity matrix) and then

$$Q_{n+1}(t) = \int_0^t C(\tau) Q_n(\tau)\, d\tau$$

then

$$\exp(-A)\exp(A + sB) = \sum_{n=0}^{\infty} s^n Q_n(1) \quad ; \qquad (18.24)$$

this series converges for every s.

In particular, the directional derivative of exp at the point A in the direction B is

$$\frac{d}{ds}\exp(A+sB)|_{s=0} = \exp(A) Q_1(1) = \int_0^1 \exp((1-\tau)A) B \exp(\tau A)$$

(Hint: Use the exercise [1T1] with $Y(t, s) = \exp(-tA)\exp(t(A + sB))$ and then set $t = 1$.)

Solution 1. *[1N1]*