Exercises

E23.6 [1QR] Prerequisites:[19S].Let su fix $x_0, t_0 \in \mathbb{R}$, and a bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$, with $f(x_0) = 0$ but $f(x) > 0$ for $x \neq x_0$. We want to study the autonomous problem

$$\begin{cases} x'(t) = f(x(t)) \ , \\ x(t_0) = x_0 \ . \end{cases}$$

Note that $x \equiv x_0$ is a possible solution. Show that if, for $\varepsilon > 0$ small, [a]

$$\int_{x_0}^{x_0+\varepsilon} \frac{1}{f(y)} \, \mathrm{d}y = \infty \tag{23.7}$$

$$\int_{x_0-\varepsilon}^{x_0} \frac{1}{f(y)} \, \mathrm{d}y = \infty \tag{23.8}$$

then $x \equiv x_0$ is the only solution; while otherwise there are many class C^1 solutions: describe them all.

Solution 1. *[1QS]*

Conditions (23.7) and (23.8) are a special case of *Osgood uniqueness condition*, see Problem 2.25 in [18].

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[a] If the condition holds for a $\varepsilon > 0$ then it holds for every $\varepsilon > 0$, since $f > 0$ far from x_0.