Exercises

E23.c.5 [1RK] Discuss the differential equation

$$\begin{cases} y'(x) = \frac{1}{y(x)-x^2} \\ y(0) = a \end{cases}$$

for $a \neq 0$, studying in a qualitative way the existence (local or global) of solutions, and the properties of monotonicity and convexity/concavity. [a]

Show that the solution exists for all positive times.

Show that for $a > 0$ the solution does not extend to all negative times.

Difficulty:*. Show that there is a critical $\tilde{a} < 0$ such that, for $\tilde{a} < a < 0$ the solution does not extend to all negative times, while for $a \leq \tilde{a}$ the solution exists for all negative times; also for $a = \tilde{a}$ you have $\lim_{x\to-\infty} y(x) - x^2 = 0$.

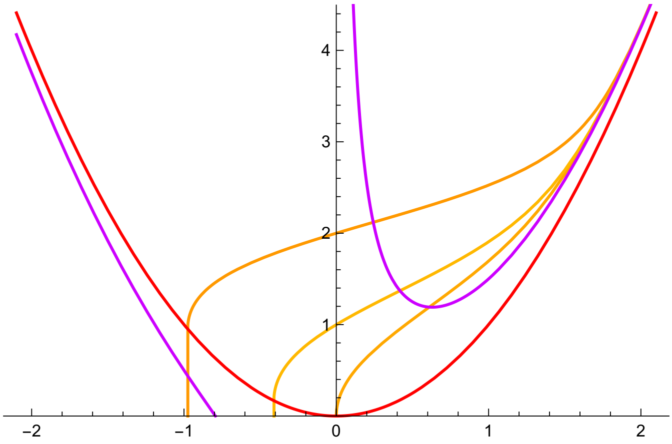

Figure 9: Exercise 23.c.5. Solutions for $a > 0$

In purple the line of inflections. In red the parabola where the derivative of the solution is infinite. In yellow the solutions with initial data $y(0) = 2$, $y(0) = 1$, $y(0) = 1/1000$.

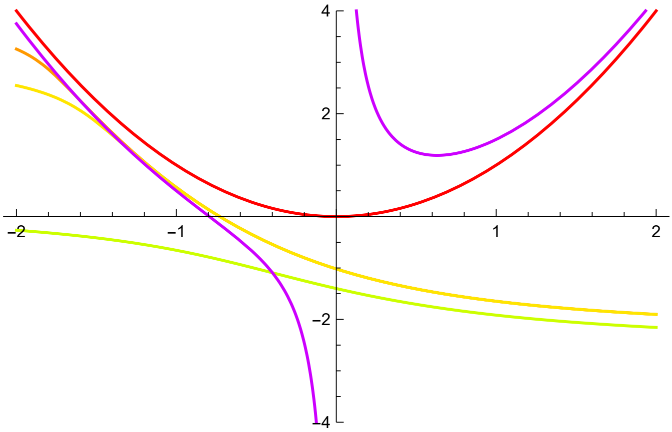

Figure 10: Exercise 23.c.5. Solutions for $a < 0$

In purple the line of inflections. In red the parabola where the derivative of the solution is infinite. Solutions are drawn with initial data $a = -1.4$ ("green"), $a = -1.0188$ ("orange") and $a = -1.019$ ("yellow"). Note that the latter two differ only by 0.0002 in their initial data (indeed they are indistinguishable in the graph for $x > -1$), but then for $x < -1$ they move apart quickly, and for $x = -2$ they are respectively 3.25696 and 2.54856, with a difference of about 0.7 !

Solution 1. *[1RP]*

[a] The differential equation is taken from exercise 13 in [**?**].