Exercises

E24.40 [1T1] Prerequisites:[118],[11K].Difficulty:*.

Let $V = \mathbb{C}^{n\times n}$ a matrix space, we equip it with a submultiplicative norm $\|C\|_V$. Let $C \in V$ and let $A, B : \mathbb{R} \to V$ be continuous curves in space of matrices.

- We recursively define $Q_0 = C$, and

$$Q_{n+1}(s) = \int_0^s A(\tau)Q_n(\tau)B(\tau)\,\mathrm{d}\tau \quad ;$$

show that the series

$$Y(t) = \sum_{n=0}^{\infty} Q_n(t)$$

is well defined, showing that, for every $T > 0$, it converges totally in the space of continuous functions $C^0 = C^0([-T, T] \to V)$, endowed with the norm

$$\|Q\|_{C^0} \stackrel{\text{def}}{=} \max_{|t|\le T} \|Q(t)\|_V \quad .$$

- Show that the function just defined is the solution of the differential equation

$$\frac{d}{dt} Y(t) = A(t)Y(t)B(t) \quad , \quad Y(0) = C \ .$$

- If A, B are constant, note that

$$Y(t) = \sum_{n=0}^{\infty} t^n \frac{A^n C B^n}{n!} \quad .$$

Solution 1. [1T2]