Exercises

E24.1 [1T9] Note:*reworked from the written exam held January 26th, 2016.*

Let $(q_n)_{n \geq 1}$ be an enumeration of the rationals of $(0, 1)$ and define

$$f(t) \stackrel{\text{def}}{=} \sum_{n \,:\, q_n < t} 2^{-n}$$

and

$$g(t) \stackrel{\text{def}}{=} \sum_{n \,:\, q_n \leq t} 2^{-n}$$

for $t \in (0, 1)$.

- Show that f, g are strictly increasing.
- Calculate limits for $t \downarrow 0$ and $t \uparrow 1$.
- Show that f is left continuous, g is right continuous, and that

$$\lim_{\tau \to t+} f(\tau) = g(t) \quad , \quad \lim_{\tau \to t-} g(\tau) = f(t) \quad .$$

- Also show that f is discontinuous in t if and only if $t \in \mathbb{Q} \cap (0, 1)$; and similarly for g.
- What changes if we replace 2^{-n} with the term a_n of an absolutely convergent series?

Solution 1. [1TB]