Exercises

E24.1 [1TY] Note: *written exam 12/1/2013.*

Given a subset E of $\mathbb{N}$ and an integer $n \in \mathbb{N}$, the expression

$$\frac{\mathrm{card}(E \cap \{0, 1, ..., n\})}{n+1}$$

indicates which fraction of the segment $\{0, 1, ..., n\}$ is contained in E. The notion of "density" in $\mathbb{N}$ of E refers to the behavior of such fractions as n tends to infinity. Precisely, we define the upper density $\overline{d}(E)$ of E and its lower density $\underline{d}(E)$ as

$$\overline{d}(E) = \limsup_{n\to\infty} \frac{\mathrm{card}(E \cap \{0, 1, ..., n\})}{n+1} \quad ,$$

$$\underline{d}(E) = \liminf_{n\to\infty} \frac{\mathrm{card}(E \cap \{0, 1, ..., n\})}{n+1} \quad .$$

If $\overline{d}(E) = \underline{d}(E) = d \in [0, 1]$, E is said to have density d. (See also [52].)

(a) Prove that, for every $\alpha \in \mathbb{R}, \alpha \geq 1$, the set $E_\alpha = [n\alpha] : n \in N$ has density $d = 1/\alpha$ (the symbol $[x]$ indicates the integer part of $x \in R$).

(b) Let $E = \{m_0, m_1, ..., m_k, ...\}$ be an infinite set, with $m_0 < m_1 < ... < m_k < ....$ Prove that $\overline{d}(E) = \limsup_{k\to\infty} \frac{k}{m_k}$ and $\underline{d}(E) = \liminf_{k\to\infty} \frac{k}{m_k}$.

(c) Find a set E with $\overline{d}(E) = \overline{d}(\mathbb{N} \setminus E) = 1$.