Exercises

E24.16 [1V4]Topics:matrix,determinant.Prerequisites:[1V2].Difficulty:*.

We want to generalize the results of the previous exercise [1V0] to the case of matrices $n \times n$.

Recall the following properties of the determinant of matrices $A \in \mathbb{R}^{n\times n}$.

- The rank is the dimension of the image of A (considered as a linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$) and is also the maximum number of linearly independent columns in A.
- A has rank n if and only $\det(A) \neq 0$.
- If you exchange two columns in A, the determinant changes sign;
- if you add a multiple of another column to a column, the determinant does not change.
- The characterization of rank through minors, *"The rank of A is equal to the highest order of an invertible minor of A"*.
- Laplace's expansion of the determinant, and Jacobi's formula (*cf* [1V2]).
- The determinant of A is equal to the determinant of the transpose; So every previous result holds, if you read "row" instead of "column".

See also in [72, 58].

Show the following results.

1. Show that the gradient of the function $\det(A)$ is not zero, if and only if the rank of A is at least $n-1$.
2. Let Z be the set of matrices $\mathbb{R}^{n\times n}$ with null determinant. Show that it is a closed set with an empty interior.
3. Fix B a matrix with rank at most $n-2$, show that the thesis of the theorem is false in the neighborhoods U_B of the matrix B, in the sense that $Z \cap U_B$ is not contained in a surface[a].

[[1V5]]

Solution 1. *[1V6]*

[a] This problem is simpler than you think... There are too many matrices with zero determinant close to B...