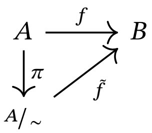
[20Q]

- Suppose we have a function $f : A \to C$; we will say that this is **invariant** (or "**compatible with** $\sim$") if

$$\forall x, y \in A, \quad x \sim y \Rightarrow f(x) = f(y) \quad . \tag{2.193}$$

  This implies that f is constant on each equivalence class; so f **passes to the quotient**, that is, there is a well-defined function $\tilde{f} : {}^{A}/_{\sim} \to B$ such that $\tilde{f}([x]) = f(x)$ for each $x \in A$; i.e. $\tilde{f} \circ \pi \equiv f$.



- Similarly we will proceed if f is a multi-argument function $f : A_1 \times A_2 \times \dots A_n \to C$, and on one or more of these sets A_1 there are equivalence relations: in this case we can pass the associated variables to the quotients. For example when there is an equivalence relation $\sim$ on A_1, we will require that

$$\forall x, y \in A_1, \forall a_2 \in A_2 \dots \forall a_n \in A_n \dots \quad x \sim y \Rightarrow f(x, a_2, \dots a_n) =$$

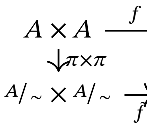
  and then we can *switch to the quotient* and define the function $\tilde{f} : {}^{A_1}/_{\sim} \times A_2 \times \dots A_n \to C$ so that

$$\tilde{f}(\pi(x), a_2, \dots a_n) = f(x, a_2, \dots a_n) \quad .$$

- Similar reasoning can be made for relation $R \in A \times B$; formally we can go back to the previous case, thinking of R as a function that has domain $A \times B$ and the set {"true"," false"} as image; more explicitly, we will say that R is invariant with respect to the relation $\sim$ on A if

$$\forall x, y \in A, \forall b \in B \quad x \sim y \Rightarrow (xRb \Leftrightarrow yRb) \quad ;$$

  and in this case we can define the relation $\tilde{R}$ "*projected to quotient*" between ${}^{A}/_{\sim}$ and B.



- In some cases a function can be *projected to the quotient* in domain and codomain; we present a simple case, with two variables, that will be used in the following. Consider a function $f : A \times A \to A$; it is invariant if

$$\forall x, \tilde{x}, y, \tilde{y} \in A, \quad (x \sim \tilde{x} \wedge y \sim \tilde{y}) \Rightarrow f(x, y) \sim f(\tilde{x}, \tilde{y}) \quad ; \tag{2.194}$$

  then f can be **projected to the quotient**, that is, the function

$$\tilde{f} : {}^{A}/_{\sim} \times {}^{A}/_{\sim} \to {}^{A}/_{\sim}$$

  is well defined when abiding to the rule

$$\forall x, y \in A \quad \tilde{f}([x], [y]) = [f(x, y)] \quad .$$