

**Exercise 6.6.** [20W] Prerequisites: [20V]. Difficulty: \*. Having fixed  $\alpha > 1$ , we define, for  $x \in \mathbb{R}$ ,

$$\alpha^x = \sup\{\alpha^p : p \in \mathbb{Q}, p \leq x\} \quad ;$$

show that:

- this is a good definition (i.e. that the set on the right is bounded above and not empty).
- If  $x$  is rational then  $\alpha^x$  (as above defined) coincides with the definition in the previous exercise [20V].
- show that  $x \mapsto \alpha^x$  is strictly increasing.
- Show that

$$\alpha^x \alpha^y = \alpha^{x+y} \quad , \quad (\alpha^x)^y = \alpha^{(xy)} \quad .$$

See also the exercise [21N].