Theorem 7.20. [21C] *Assume that* $a_n \neq 0$. *Let* $\alpha = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ *then*

- *if* $\alpha < 1$ *the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely;*
- *if* $\alpha \geq 1$ *nothing can be concluded.*

Proof.

- If $\alpha < 1$, taken $L \in (\alpha, 1)$ you have eventually $\frac{|a_{n+1}|}{|a_n|} < L$ so there is a N for which $\frac{|a_{n+1}|}{|a_n|} < L$ for each $n \geq N$, by induction it is shown that $|a_n| \leq L^{n-N}|a_N|$ and ends by comparison with the geometric series.
- Let's see some examples. For the two series $1/n$ and $1/n^2$ you have $\alpha = 1$.

  Defining

$$a_n = \begin{cases} 2^{-n} & n \text{ even} \\ 2^{2-n} & n \text{ odd} \end{cases} \tag{7.21}$$

  we obtain a convergent series but for which $\alpha = 2$.

□