Theorem 7.23. [21D] *If $(a_n)_n \subset \mathbb{R}$ has positive terms and is monotonic (weakly) decreasing, the series converges if and only if the series*

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

converges.

Proof. Since the sequence $(a_n)_n$ is decreasing, then for $h \in \mathbb{N}$

$$2^h a_{2^{(h+1)}} \le \sum_{k=2^h+1}^{2^{(h+1)}} a_k \le 2^h a_{2^h} \quad . \tag{7.24}$$

We note now that

$$\sum_{h=0}^{N} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \sum_{n=2}^{2^{N+1}} a_n$$

and therefore

$$\sum_{h=0}^{\infty} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \lim_{N\to\infty} \sum_{h=0}^{N} \sum_{k=2^h+1}^{2^{(h+1)}} a_k = \lim_{N\to\infty} \sum_{n=2}^{2^{(N+1)}} a_n = \sum_{n=2}^{\infty} a_n \quad .$$

so we can add the terms in (7.24) to get

$$\sum_{h=0}^{\infty} 2^h a_{2^{(h+1)}} \le \sum_{n=2}^{\infty} a_n \le \sum_{h=0}^{\infty} 2^h a_{2^h}$$

where the term on the right is finite if and only if the one on the left is finite, because

$$\sum_{h=0}^{\infty} 2^h a_{2^h} = a_1 + 2 \sum_{h=0}^{\infty} 2^h a_{2^{(h+1)}} \quad :$$

the proof ends by the comparison theorem □