Lemma 4.13. [27N] *(Solved on 2022-11-03)* $\forall n, h \in \mathbb{N}, f_{S(h)}(n) = S(f_h(n))$

Proof. Recall that $S(f_h(n)) = f_h(S(n))$ by recursive definition; Consider

$$P(n) \doteq \forall h \in \mathbb{N}, f_{S(h)}(n) = S(f_h(n)) \quad ;$$

$P(0)$ is the clause

$$\forall h \in \mathbb{N}, f_{S(h)}(0) = S(f_h(0)) = S(h)$$

which is true because it is the initial value of the recursive definition $f_{S(h)}$ and f_h. For the inductive step we assume that $P(n)$ is true and we study $P(S(n))$, within which we can say

$$f_{S(h)}(S(n)) \stackrel{(1)}{=} S(f_{S(h)}(n)) \stackrel{(2)}{=}$$
$$SS(f_h(n)) \stackrel{(3)}{=} S(f_h(S(n))$$

where in (1) we used the recursive definition of f_h with $S(h)$ instead h, in (2) we used the inductive hypothesis, and in (3) we used the recursive definition of f_h. This completes the inductive step.
(Note, in the first step, how important it is that in the definition of $P(n)$ there is $\forall h \in \mathbb{N}, \ldots$). □