Proposition 4.48. [28R]

- (Addition and ordering compatibility) *You have $n \leq m$ if and only if $n + k \leq m + k$.*
- (Multiplication and ordering compatibility) *When $k \neq 0$ you have $n \leq m$ if and only if $n \times k \leq m \times k$.*

In particular (remembering [28M]) the map $n \mapsto n \times h$ is strictly increasing (and hence injective).

Proof. We will use some properties left for exercise.

- If $n \leq m$, by definition $m = n + h$, then $n + k \leq m + k$ because $m + k = n + h + k$ (note that we are using associativity). If $n + k \leq m + k$ let then j the only natural number such that $n + k + j = m + k$ but then $n + j = m$ by cancellation [27V].
- If $n \leq m$ then $m = n + h$ therefore $m \times k = n \times k + h \times k$ so $n \times k \leq m \times k$. Vice versa let $k \neq 0$ and $n \times k \leq m \times k$ i.e. $n \times k + j = m \times k$: divide j by k using the division [28J], we write $j = q \times k + r$ therefore for associativity $(n + q) \times k + r = m \times k$ but for the uniqueness of the division $r = 0$; eventually collecting $(n + q) \times k = m \times k$ and using [28M] we conclude that $(n + q) = m$.

$\square$