Remark 2.3. *The parentheses symbols () are unfortunately quite overloaded in common Mathematical language.* [2FG]

- *They are used to group algebraic operations, to induce a different order of operations (wrt the standard rules of precedence). For example, for $x, y \in \mathbb{R}$,* [†1] *the expression $x(y+2)$ is identical to $xy + 2x$ and not to $xy + 2$.*
- *They are used to denote arguments of functions. For example the expression $f(x+y)$ should be read as $fx + fy$, if $f, x, y \in \mathbb{R}$* [†2]*; whereas, if f is a function $f : \mathbb{R} \to B$, then $f(x+y)$ is the result $f(z)$ obtained by evaluating f on the element $z = x + y$.*

  To distinguish these two usages, it may be sufficient to add an explicit symbol to denote "multiplication", writing $f * (x+y)$ when it should be read as $f * x + f * y$. (Some authors also write $f.(x+y)$ with a "dot")
- *They are used to define intervals, for example, $(1, \pi)$ may be shorthand for:* «the set of real numbers larger than 1 and smaller than π;» *in formula*

  $$(1, \pi) = \{t \in \mathbb{R} : 1 < t < \pi\} ;$$

  this extends to ordered sets, see Sect. [2DW].
- *They are used to represent elements of the Cartesian product; for example, $(1, \pi)$ is point in $\mathbb{R}^2$ with 1 as abscissa and π as ordinate.*

While the first and second situations are usually discernable and recognizable, the third and fourth can cause confusion.

Some care is needed in parsing statements involving Cartesian products of ordered sets, such as: «a point (x, y) in the rectangle R of the plane that is the product $R = (0, 1) \times (2, 4)$». *Here (x, y) is a point in $\mathbb{R}^2$ whereas $(0, 1), (2, 4)$ are intervals in $\mathbb{R}$.*

To avoid confusion, we may use a different notation for points and/or for intervals: many symbols that are similar to "parentheses" are available nowadays in the extended Unicode codespace, and are available to LaTeX users through the `unicode-math` package.

For example, in the above statement, we may use this (non-standard) notation: use barred parentheses $⦇\ldots⦈$ to denote the point in $\mathbb{R}^2$ with x as abscissa and y as ordinate; use double parentheses $⸨a, b⸩ = \{t \in \mathbb{R} : a < t < b\}$ for intervals; so as to obtain «a point $⦇x, y⦈$ in the rectangle R of the plane that is the product $R = ⸨0, 1⸩ \times ⸨2, 4⸩$». *In this case, for typographic consistency, we may use at the same time double brackets for closed-ended intervals, such as $[\![2, 4]\!]$.*

This may be considered overkill for this example. But the situation can be more complicated, though!

For example, we may be dealing with intervals of elements of an ordered set X, that is also a Cartesian product $X = X_1 \times X_2$ of ordered sets X_1, X_2 (!) [†3] *In that case, we should first label the orders, for example: $\leq_1$ being the order relation on X_1, $\leq_2$ being the order relation on X_2, and $\preceq$ being the order relation on X; and use a (non-standard) notation for intervals, such as*

$$⸨a, b⸩_1 = \{t \in X_1 : a <_1 t <_1 b\}$$

[†1] Or, more in general, if x, y are elements of a *ring* where multiplication is denoted by justaposition of symbols.

[†2] Again, more in general, if f, x, y are elements of a *ring* where multiplication is denoted by justaposition.

[†3] BTW, there is a standard method to order a Cartesian product of ordered sets, see Sect. [2FH].

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for open-ended intervals in the first set (with extremes $a, b \in X_1$),

$$(\!(z, w)\!)_{\preceq} = \{x \in X : w \prec x \prec z\}$$

for open-ended intervals in the Cartesian product X (with extremes $z, w \in X$), and so on. Again, for typographic consistency, we may use double brackets for closed-ended intervals, such as

$$[\![a, b]\!]_1 = \{x \in X_1 : a \leq_1 x \leq_1 b\}$$

and so on.

In the following we will often use the usual parentheses () ; but in certain contexts we will use the notation proposed in this note (when it could help in understanding the text).

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